NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for **TMA4130 Matematikk 4N**

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Examination date: 2. December 2014

Examination time (from–to): 09:00–13:00

Permitted examination support material: Code C: Basic calculator. Rottmann: *Mathematical formulæ.*

Other information:

- All answers have to be justified, and they should include enough details in order to see how they have been obtained.
- All 10 sub-problems carry the same weight for grading.

Language: English

Number of pages: 3

Number pages enclosed: 2

Checked by:

Date Signature

Problem 1 Let $f(x)$ the 2π-periodic function defined by

$$f(x) = \begin{cases} -\sin(x) & \text{if } -\pi < x \leq 0, \\ \sin(x) & \text{if } 0 < x \leq \pi. \end{cases}$$

a) Compute the Fourier series of the function $f(x)$.

b) Use the Fourier series of the function $f(x)$ computed in exercise **a)** to find the value of the serie

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$$

Problem 2 Find the inverse Laplace transform of the function

$$F(s) = \frac{11 - s}{s^2 - 2s - 3}.$$

Problem 3 Use the Laplace transform in order to solve the initial value problem

$$\begin{aligned} y'' &= y + \delta(t-1) + 2e^{t-1}u(t-1), \\ y(0) &= 0, \\ y'(0) &= 1. \end{aligned}$$

Problem 4

a) Find all the non-trivial solutions of the heat equation

$$\frac{\partial u}{\partial t} = 2 \cdot \frac{\partial^2 u}{\partial x^2} \qquad \text{where} \qquad 0 \leq x \leq \pi \qquad \text{and} \qquad t \geq 0$$

that are of the form $u(x,t) = F(x) \cdot G(t)$, and that satisfy the boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 0 \qquad \text{and} \qquad u(\pi, t) = 0 \qquad \text{for } t \geq 0.$$

b) With the results from **a)** find the solutions that additionally satisfy the initial condition

$$u(x,0) = \cos(x)\cos\left(\frac{3}{2}x\right) \qquad \text{for} \qquad 0 \leq x \leq \pi\,.$$

Problem 5

a) Find the polynomial of lowest degree that interpolate the points

x_i	-2	1	0	1	2
$f(x_i)$	0	2	0	1	0

b) We want to numerically evaluate the integral

$$\int_1^3 x^2 \ln(x) dx$$

with the Simpson method such that the approximation error is smaller than 0.001. What is the largest value of the step size h that we can choose such that this accuracy is guaranteed? Use this h to compute a numerical approximation of the above integral by the Simpson method. (Use only 4 decimals in your computations)

Problem 6 Let $\mathcal{R}$ be the region defined by the lines

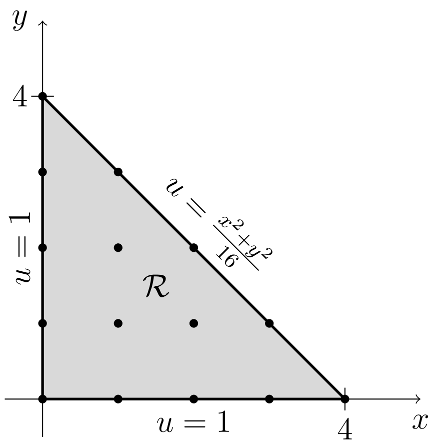
$$L_1 : y = 4 - x \qquad L_2 : y = 0 \qquad L_3 : x = 0\,.$$

Let $u(x, y)$ be the function defined in $\mathcal{R}$ satisfying the Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2xy$$

and the boundary conditions

$$\begin{aligned} u(x,y) &= \tfrac{x^2+y^2}{16} && \text{if } (x,y) \in L_1 \\ u(x,y) &= 1 && \text{if } (x,y) \in L_2 \\ u(x,y) &= 1 && \text{if } (x,y) \in L_3 \end{aligned}$$



a) Let us define the points $(x_i, y_j) = (i \cdot h, j \cdot h)$ with $h = 1$. Use the method of difference equations with $h = 1$ in order to set up a linear system for finding approximations of the values $u(1,1)$, $u(1,2)$ and $u(2,1)$.

b) Write the linear system you have found in **a)** in such a form that you can apply the Gauss-Seidel method. Then carry out three iterations of the Gauss-Seidel method with starting values 0 in all the variables. (Use only 4 decimals in your computations.)

Fourier

$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\, d\omega$	$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$
$f * g(x)$	$\sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)$
$f'(x)$	$i\omega\hat{f}(\omega)$
e^{-ax^2}	$\frac{1}{\sqrt{2a}} e^{-\omega^2/4a}$
$e^{-a\lvert x\rvert}$	$\sqrt{\frac{2}{\pi}} \frac{a}{\omega^2+a^2}$
$\frac{1}{1+x^2}$	$\sqrt{\frac{\pi}{2}} e^{-\lvert\omega\rvert}$
$f(x) = 1$ for $\lvert x\rvert < a$, 0 otherwise	$\sqrt{\frac{2}{\pi}} \frac{\sin \omega a}{a}$

Laplace transform

$f(t)$	$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$
$f'(t)$	$sF(s) - f(0)$
$tf(t)$	$-F'(s)$
$e^{at} f(t)$	$F(s-a)$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$f(t-a)u(t-a)$	$e^{-sa}F(s)$
$\delta(t-a)$	e^{-as}

Numerics

- Newton's method: $x_{k+1} = x_k - f(x_k)/f'(x_k)$.
- Lagrange interpolation polynomial: $L_k(x) = \frac{(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$,
  $p_n(x) = \sum_{k=0}^{n} L_k(x) f(x_k)$
- Trapezoid rule: $\int_a^b f(x)\,dx \approx h\Big[\frac{1}{2}f(a) + f(x_1) + f(x_2) + \ldots + f(x_{n-1}) + \frac{1}{2}f(b)\Big]$

  Error of the trapezoid rule: $|\epsilon| \le h^2 \frac{b-a}{12} \max_{a\le x\le b} |f''(x)|$.
- Simpson rule: $\int_a^b f(x)\,dx \approx \frac{h}{3}\Big[f_0 + 4f_1 + 2f_2 + 4f_3 + \ldots + 2f_{n-2} + 4f_{n-1} + f_n\Big]$ with $f_i = f(x_i)$.

  Error of the Simpson rule: $|\epsilon| \le h^4 \frac{b-a}{180} \max_{a\le x\le b} |f^{(4)}(x)|$.
- Gauss–Seidel iteration: $\mathbf{x}^{(k+1)} = \mathbf{b} - \mathbf{L}\mathbf{x}^{(k+1)} - \mathbf{U}\mathbf{x}^{(k)}$ with $\mathbf{A} = \mathbf{I} + \mathbf{L} + \mathbf{U}$.
- Jacobi iteration: $\mathbf{x}^{(k+1)} = \mathbf{b} + (\mathbf{I} - \mathbf{A})\mathbf{x}^{(k)}$
- Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_n, \mathbf{y}_n)$
- Improved Euler method: $\mathbf{k}_1 = h\mathbf{f}(x_n, \mathbf{y}_n)$, $\mathbf{k}_2 = h\mathbf{f}(x_n + h, \mathbf{y}_n + \mathbf{k}_1)$,
  $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}\mathbf{k}_1 + \frac{1}{2}\mathbf{k}_2$.
- Classical Runge–Kutte method:
  $\mathbf{k}_1 = h\mathbf{f}(x_n, \mathbf{y}_n)$, $\mathbf{k}_2 = h\mathbf{f}(x_n + h/2, \mathbf{y}_n + \mathbf{k}_1/2)$,
  $\mathbf{k}_3 = h\mathbf{f}(x_n + h/2, \mathbf{y}_n + \mathbf{k}_2/2)$, $\mathbf{k}_4 = h\mathbf{f}(x_n + h, \mathbf{y}_n + \mathbf{k}_3)$,
  $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{6}\mathbf{k}_1 + \frac{1}{3}\mathbf{k}_2 + \frac{1}{3}\mathbf{k}_3 + \frac{1}{6}\mathbf{k}_4$.
- Backward Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_{n+1}, \mathbf{y}_{n+1})$
- Finite differences: $\frac{\partial u}{\partial x}(x,y) \approx \frac{u(x+h,y) - u(x-h,y)}{2h}$
  $\frac{\partial^2 u}{\partial x^2}(x,y) \approx \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2}$
  $\frac{\partial u}{\partial y}(x,y) \approx \frac{u(x,y+h) - u(x,y-h)}{2h}$
  $\frac{\partial^2 u}{\partial y^2}(x,y) \approx \frac{u(x,y+h) - 2u(x,y) + u(x,y-h)}{h^2}$