NTNU

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TMA4125 Matematikk 4N

Fourier Transforms: Continuous and Discrete.

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February 14, 2024

Fourier for non-periodic functions?

Question.
How can we analyse functions $f : \mathbb{R} \to \mathbb{C}$ that are not periodic?

Idea.

- For some L "cut out" the interval $[-L, L]$
- assume that it is periodic and use Fourier series
- Let $L \to \infty$

Definition: (Continuous) Fourier Transform

Let $f \in L_1(\mathbb{R})$ (absolute integrable, i. e. $\int_{-\infty}^{\infty} |f(x)| \, \mathrm{d}x < \infty$).

Then the (continuous) Fourier Transform is defined by

$$\hat{f}(\omega) := \mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-\mathrm{i}\omega x} \, \mathrm{d}x, \qquad \omega \in \mathbb{R}.$$

If $g(\omega)$ is absolutely integrable, then the inverse Fourier transform is defined by

$$\check{g}(x) := \mathcal{F}^{-1}(g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) \mathrm{e}^{\mathrm{i}\omega x} \, \mathrm{d}\omega, \qquad x \in \mathbb{R}.$$

Be careful with the definition of the Fourier Transform

Warning. It is not uniquely given, where to place the factor $\frac{1}{2\pi}$
The following definitions are usually used

- $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-\mathrm{i}\omega x} \, \mathrm{d}x$ (ours, both $\mathcal{F}$ and $\mathcal{F}^{-1}$ with same factor)
- $\int_{-\infty}^{\infty} f(x) \mathrm{e}^{-\mathrm{i}\omega x} \, \mathrm{d}x$ (then the inverse has $\frac{1}{2\pi}$ as a factor)
- $\frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-\mathrm{i}\omega x} \, \mathrm{d}x$ (no factor for the inverse)
- $\int_{-\infty}^{\infty} f(x) \mathrm{e}^{-2\pi\mathrm{i}\omega x} \, \mathrm{d}x$ (frequency in Hz, no factor for the inverse)

Sometimes even the minus in the exponent might be with the inverse transform.

Be careful. When you see a Fourier transform and first check which definition was used.

Cosine & Sine Transform

Using Eulers formula we can do the same derivation as before using $\cos$ and $\sin$.

Let f, g be even functions and $f, g \in L_1(\mathbb{R})$.
We define the Fourier Cosine Transform (and its inverse) as

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(\omega x)\, \mathrm{d}x \qquad \check{g}_c(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(\omega) \cos(\omega x)\, \mathrm{d}\omega$$

Let f, g be odd functions and $f, g \in L_1(\mathbb{R})$.
We define the Fourier Sine Transform (and its inverse) as

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(\omega x)\, \mathrm{d}x \qquad \check{g}_s(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(\omega) \sin(\omega x)\, \mathrm{d}\omega$$

Advantage. If f (and g) are real-valued, so are their transforms.
Warning. Again – be careful with the scaling.

Inversion of the Fourier Transform

Theorem. Let f and $\hat{f}$ be absolutely integrable.

Then

$$f(x) = \mathcal{F}^{-1}(\hat{f}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) \mathrm{e}^{\mathrm{i}\omega x} \, \mathrm{d}\omega$$

Proof. Omitted.

Time vs Frequency Domain

When we consider f and its Fourier Transform $\hat{f}$ we have

$$f(x) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} \hat{f}(\omega)$$

and we call

- f to be in the time domain
- $\hat{f}$ to be in the frequency domain

Example I

Example. Let $a, b \in \mathbb{R}$, $a < b$. We consider the indicator function

$$f(x) = X_{[a,b]} = \begin{cases} 1 & \text{if } x \in [a, b] \\ 0 & \text{else.} \end{cases}$$

Examples II

Example. Let $a > 0$ be given and set $f(x) = \mathrm{e}^{-a|x|}$. Then

$$\mathcal{F}(\mathrm{e}^{-a|x|}) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + \omega^2}$$

Example. Let $a > 0$ be given and set $f(x) = \mathrm{e}^{-ax^2}$. Then

$$\mathcal{F}(\mathrm{e}^{-ax^2}) = \frac{1}{\sqrt{2a}} \mathrm{e}^{-\frac{\omega^2}{2a}}$$

or in other words:
The Fourier transform of a Gaussian is again a Gaussian.

Linearity & Derivatives and the Fourier transform

Theorem. Let $a, b \in \mathbb{C}$ and $f, g \in L^1(\mathbb{R})$. Then

$$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$$

Proof. Left as an exercise, but follows directly from linearity of integration

Theorem. Assume that both f and f' are in $L^1(\mathbb{R})$ and that $f(x) \to 0$ for $x \to \pm\infty$.
Then it holds that

$$\mathcal{F}(f') = i\omega\mathcal{F}(f).$$

Proof.

Fourier Transform and Convolution

Definition. (Convolution II, Fourier version)
Let $f, g \in L^1(\mathbb{R})$. Then the convolution $f * g$ is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y) g(x - y) \, \mathrm{d}y.$$

This is actually well defined and we even have $f * g \in L^1(\mathbb{R})$.

Theorem. Similar to the 2π periodic case we have $f * g = g * f$.

Theorem. Let $f, g \in L^1(\mathbb{R})$. Then

$$\mathcal{F}(f * g) = \sqrt{2\pi} \cdot \mathcal{F}(f) \cdot \mathcal{F}(g).$$

So again. A convolution in time turns into a multiplication in frequency domain.

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The Discrete Fourier Transform

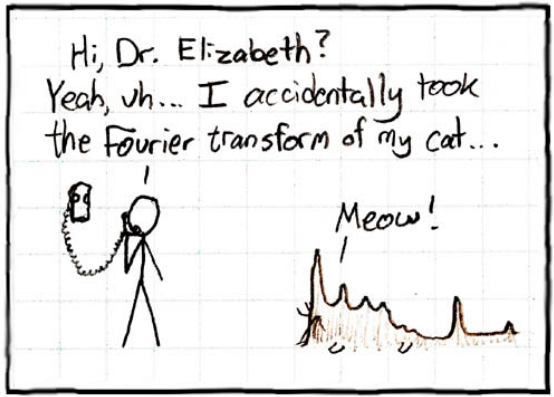
Overview and Motivation

Concerning Fourier Transforms we now have Transforms for

1. 2π-periodic functions $f \Rightarrow$ Fourier Series $c_k(f), k \in \mathbb{Z}$
2. functions $f \in L^1(\mathbb{R})$ we obtain the Fourier Transform $\hat{f}(\omega)$

Question. Can we Fourier transform a Signal $(f_0, f_1, \ldots, f_{N-1})$?

(or: what else can we Fourier transform?)



source:https://xkcd.com/26/

Towards Signals: Discretizing Fourier coefficients

Idea. Use a composite trapezoidal rule to approximate $c_k(f)$.

We use $[0, 2\pi)$ and a 2π-periodic function f.

We choose N equidistant sampling points $x_j = \dfrac{2\pi j}{N}, \quad j = 0, \dots, N-1.$

Then we obtain for any $k \in \mathbb{Z}$.

$$c_k(f) = \frac{1}{2\pi} \int_0^{2\pi} f(x) \mathrm{e}^{-\mathrm{i}kx} \, \mathrm{d}x \approx \frac{1}{N} \sum_{j=0}^{N-1} f\Big(\frac{2\pi j}{N}\Big) \mathrm{e}^{-2\pi \mathrm{i} jk/N}$$

Shift view: Instead of the function f
$\Rightarrow$ think of a given signal (or vector) of values

$$\big(f_0, f_1, \dots, f_{N-1}\big) \in \mathbb{C}^N$$

here:

$$f_j \coloneqq f\Big(\frac{2\pi j}{N}\Big), \quad j = 0, \dots, N-1.$$

The Discrete Fourier Transform

For a given signal $\boldsymbol{f} = (f_0, f_1, \ldots, f_{N-1})^{\mathrm{T}} \in \mathbb{C}^N$
the Discrete Fourier Transform (DFT) is defined as

$$\hat{f}_k = \frac{1}{N} \sum_{j=0}^{N-1} f_j \, \mathrm{e}^{-2\pi \mathrm{i} jk/N}, \qquad k = 0, \ldots, N-1.$$

We introduce $w_N = \mathrm{e}^{-2\pi \mathrm{i}/N}$ and $\hat{\boldsymbol{f}} = (\hat{f}_0, \ldots, \hat{f}_{N-1})^{\mathrm{T}} \in \mathbb{C}^N$.
Then we write the DFT also in matrix-vector form

$$\hat{\boldsymbol{f}} = \frac{1}{N} \mathcal{F}_N \boldsymbol{f}$$

where the matrix $\mathcal{F}_N \in \mathbb{C}^{N \times N}$ is given by

$$\mathcal{F}_N = \left(\mathrm{e}^{-2\pi \mathrm{i} jk/N} \right)_{j,k=0}^{N-1} = \left(w_N^{jk} \right)_{j,k=0}^{N-1}$$

is the Nth Fourier matrix. We can easily see $\mathcal{F}_N = \mathcal{F}_N^{\mathsf{T}}$

DFT – Historical Remarks

The discrete Fourier transform is even older than the theory developed by Fourier (1807)

- mentioned by Lagrange 1758 for sine functions (DST)
- Lagrange, Clairaut (1754), and Euler used them to determine orbits of celestial bodies
- was also used by Gauss to determine the orbit of Ceres ($N = 12$, $\approx$ in 1801)

Properties of the discrete Fourier coefficients $\hat{f}_k$

N-periodicity The discrete Fourier coefficients are N-periodic, i. e.

$$\hat{f}_k = \hat{f}_{k+N}, \qquad k \in \mathbb{Z}.$$

(Exercise, note what happens to w_N^N).

Symmetry for real-valued signals if the f_j (or the function f) is real-valued then

$$\hat{f}_k = \overline{\hat{f}_{-k}}$$

Aliasing formula. Let $f \in C([-\pi, \pi))$ and let $\sum_k |c_k(f)| < \infty$ then the Aliasing formula holds:

$$\hat{f}_k = \sum_{\ell \in Z} c_{k+\ell N}(f), \quad k \in \mathbb{Z}.$$

Results of Aliasing

For even N and if f is a trigonometric polynomial of degree $\frac{N}{2}$, i. e. of the form

$$f(x) = \sum_{k=-N/2+1}^{N/2-1} c_k(f) \mathrm{e}^{2\pi \mathrm{i} k x}$$

in other words: in the sums only the terms $\ell = 0$ are present

Then $\hat{f}_k = c_k(f)$, $k = -N/2+1, ..., N/2-1$.

Remark.
This also means: Given $f_0, \ldots, f_{N-1}$ sampling values of such an f, we can uniqely reconstruct the function.
Having "exactly enough samples", $2m+1$ for highest frequency $m = N/2$ is called the Nyquist rate.

This is also called periodic interpolation or trigonometric interpolation.

Inverse Discrete Fourier Transform

Theorem. For a given signal $\hat{\boldsymbol{f}} = (\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_{N-1})^{\mathrm{T}} \in \mathbb{C}^N$ the Inverse Discrete Fourier Transform (IDFT) is given by

$$f_j = \sum_{k=0}^{N-1} \hat{f}_k \, \mathrm{e}^{2\pi \mathrm{i} jk/N}, \qquad j = 0, \ldots, N-1.$$

Remember $w_N = \mathrm{e}^{-2\pi \mathrm{i}/N}$ and $\boldsymbol{f} = (f_0, \ldots, f_{N-1})^{\mathrm{T}} \in \mathbb{C}^N$.
Then we have since $\mathrm{e}^{2\pi \mathrm{i} jk/N} = \overline{w_N^{jk}}$ that

$$\boldsymbol{f} = \overline{\mathcal{F}_N} \hat{\boldsymbol{f}}$$

Proof. *Idea.* We have to show (plugging in $\hat{\boldsymbol{f}} = \mathcal{F}_N \boldsymbol{f}$) that $\boldsymbol{f} = \frac{1}{N} \overline{\mathcal{F}_N} \mathcal{F}_N \boldsymbol{f}$ or in other words that $\frac{1}{N} \overline{\mathcal{F}_N} \mathcal{F}_N$ is the identity matrix.

The Fast Fourier Transform – Motivation

Disadvantage of the DFT: Computing it takes $\mathcal{O}(N^2)$ operations.

Example.
In processing audio: Filter (e. g. remove noise) is a convolution , but 44.1kHz Sampling in Audio $\Rightarrow N = 44100$ for 1 sec. of audio data.

Audio length	Operations	(sec.) on modern CPU (i9)
1 sec.	$1.94 \cdot 10^9$	$0.388 \cdot 10^{-4}$
1 min.	$7 \cdot 10^{12}$	0.14
1 hr	$2.52 \cdot 10^{16}$	504 (8 Min.)

That is not feasible, e. g. when we need this “in real-time” (below $\frac{1}{50}$th of a second).

Can we maybe use the structure of $\mathcal{F}$ to make it faster?

The Fast Fourier Transform – Approach

Cooley & Tuckey (1965) presented an algorithm for the DFT (and hence IDFT) that we illustrate for the case for $N = 2^n$

The Radix-2-FFT.
We will just present the idea.

1. For each k computing $\hat{f}_k$ can be split the sum the first and second half
2. Then looking at even and odd k, each of these sets is a DFT($\frac{N}{2}$) (after N additions/subtractions and $N/2$ multiplications)

$\Rightarrow$ reduces $\mathcal{O}(N^2)$ to "2 times $\mathcal{O}(\frac{N^2}{4})$ plus $\mathcal{O}(\frac{N}{2})$ multiplications"

FFT – Number of operations

If we iterate this idea, we can do the DFT(N) (remember $N = 2^n$)

- 2 DFT($\frac{N}{2}$) and $N/2$ multiplications
- 4 DFT($\frac{N}{4}$) and $2\frac{N}{2}$ multipliations
- ...
- 2^k DFT($\frac{N}{2^k}$) and $k\frac{N}{2}$ multipliations
- ...
- $N = 2^n$ DFT(1) and $n\frac{N}{2}$ multiplications

The number of additions to combine the DFT(1)s again is nN.
$\Rightarrow$ combining these yields that the Fast Fourier transform requires $\mathcal{O}(nN) = \mathcal{O}(\log(N)N)$ operations.

Example.
The approx. 8 min. in the audio example reduce to $5 \cdot 10^{-4}$ sec.
speedup of about $8.9 \cdot 10^6$

FFT – Historical Remarks

You can use arbitrary factorisations $N = N_1 N_2$ but then the formula is more complicated

- 1965 Cooley & Tukey introduced the FFT, which made it fast and usable on computers
- 1905 the main idea of the FFT was described already by C. Runge
- 1801 Gauss (to determine the orbit of Ceres) split his DFT $N = 12$ into $N = 4$ and $N = 3$

Today there is a large variety of algorithms/splitting techniques collected within the FFTW.

Discrete Sine and Cosine Transform (DST & DCT)

For the discrete case you can use the same ideas we already saw for even (cosine) and odd (sine) transform approaches.

Advantage. A real-valued signal stays real-valued

Variants. For the discrete signal there are 4 ways of continuing a signal even/odd.
$\Rightarrow$ There exist

- 4 discrete sine transforms (DST-I to DST-IV)
- 4 discrete cosine transforms (DCT-I to DCT-VI)

Fast implementations based on the FFT idea are available here as well.
Most used: DCT-II.

Application: Image Processing and Compression: Barbara

172 kB

Application: Image Processing and Compression: Barbara

29.9 kB (17.3 %)

Application: Image Processing and Compression: Barbara

18.9 kB (11 %)

Application: Image Processing and Compression: Barbara

10 kB (5.8 %)

Application: Image Processing and Compression: Barbara

5.59 kB (3.25 %)

Application: Image Processing and Compression: Barbara

3 kB (1.74 %)