NTNU
Norwegian University of Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4125 Matematikk 4N**

Solution

Academic contact during examination:

Phone:

Examination date: May 15, 2023

Examination time (from–to): 9:00–13:00

Permitted examination support material: C.
One sheet A4 paper, approved by the department (yellow sheet, “gul ark”) with own handwritten notes.
Certain simple calculators.

Other information:

- All answers have to be justified, and they should include enough details in order to see how they have been obtained.
- Good Luck! | Lykke til! | Viel Glück!

Language: English

Number of pages: 16

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave
Originalen er:
1-sidig ☐ **2-sidig** ☒
sort/hvit ☒ **farger** ☐
skal ha flervalgskjema ☐

Date Signature

In the exam one could obtain 100 points and the exam was graded using the usual grading scheme, i.e.

A	B	C	D	E	F
100–89	88–77	76–65	64–53	52–41	40 and less

And the grades are distributed as follows

A	B	C	D	E	F	$\sum$
4 1.3 %	16 5.1 %	45 14.4 %	80 25.6 %	105 33.5 %	63 20.1 %	313

Problem 1. (Fixed-point iterations, 14 points)

We get the following Python code of two fix point iterations

```
def fix_point_iteration1(x, N):
    for n in range(N)
        x = (x+5) / (x+1)
    return x

def fix_point_iteration2(x0, N):
    for n in range(N):
        x = (3*x**2-5) / (2*x)
    return x
```

where we assume we are looking for a fix point $x^* \geq 0$.

a) The code contains two syntactic errors (i.e. the math formulae are correct). Find them and fix them.

b) Which two fix point iterations are performed?
Determine both funcions $g_1(x)$ and $g_2(x)$ used in the code as well as their (non-negative) fixed point(s).

c) How do we have to choose a non-negative starting point $x_0 \geq 0$ for the first method to converge?

d) What about the convergence of the second method?

Solution.

a) The first function is missing a `:` at the end of the for-loop, the second mixes `x0` and `x` as variables for the iterates (2 P.)

b) The functions read $g_1(x) = \frac{x+5}{x+1}$ and $g_2(x) = \frac{3x^2-5}{2x}$. (2 P.)
Since
$$x = \frac{x+5}{x+1} \Leftrightarrow x(x+1) = 5 + x \Leftrightarrow x^2 + x = 5 + x \Leftrightarrow x^2 = 5$$
and
$$x = \frac{3x^2-5}{2x} \Leftrightarrow 2x^2 = 3x^2 - 5 \Leftrightarrow -x^2 = -5 \Leftrightarrow x^2 = 5$$
both aim to compute the same fix point $x^* = \sqrt{5}$ (3 P.)

c) We need the derivative of g_1 which reads

$$g_1'(x) = \frac{1 \cdot (x+1) - (x+5) \cdot 1}{(x+1)^2} = -\frac{4}{(x+1)^2}.$$

For the method to converge, we have to find an interval $[a, b]$ on the non-negative real line, such that for all $x_0 \in [a, b]$ we have

- $g1 \in C^1([a, b])$
- $g_1([a, b]) \subset [a, b]$
- $|g_1'(x_0)| \leq L < 1$

Let's start with the second point. We need

$$|g_1'(x)| = \frac{4}{(x+1)^2} < 1 \Leftrightarrow 4 < (x+1)^2$$

Now since $x \geq 0$ we can take the square root on both sides and obtain $2 < x+1$ or $x > 1$. On the interval $(1, \infty)$ we also have that $\frac{x+5}{x+1} = 1 + \frac{4}{x+1} > 1$, so the first point also holds and the fix point iteration converges for all $x > 1$. (4 P.)

One can also manually check that indeed for any $x \geq 0$ the method converges, since $x+5 > x+1$ for any positive x and hence the first iterate is larger than 1.

d) For the second method the derivative reads

$$g_2'(x) = \frac{(6x) \cdot (2x) - (3x^2 - 5) \cdot 2}{4x^2} = \frac{12x^2 - 6x^2 + 10}{4x^2} = \frac{3x^2 + 5}{2x^2}$$

This is always larger than 1 since $3x^2 + 5 > 2x^2$, which is equivalent to $x^2 + 5 > 0$ which is always true.

So for *any* starting point $x_0 \geq 0$ this method does not converge to the fix point. (3 P.)

Problem 2. (B-Splines, 11 points)

We consider the cubic splines $\mathcal{S}_{3,\Delta}$ with knot vector $\Delta = [-1, 0, 1, 2, 3]$.

Which of the following functions is in the space $\mathcal{S}_{3,\Delta}$?
State a reason for each of the functions.

a) $f_1(x) = 3x^3$

b) $f_2(x) = \left|x - \frac{1}{2}\right|$

c) $f_3(x) = 4(x)_+^2 + 2(x-2)_+^3$, where $(y)_+ = \begin{cases} y & \text{if } y \geq 0 \\ 0 & \text{else} \end{cases}$

d) What is the dimension of the space $\mathcal{S}_{3,\Delta}$?

e) What does it mean for a spline $s(x) \in S_{3,\Delta}$ to be *natural*?

Solution.

We have to check whether for a function f

1. $f\big|_{I_j} \in \mathbb{P}_3$, $I_j = [x_{j-1}, x_j]$,
2. $f \in C^2[a,b]$,

where we here have $x_j = -1 + j$, $j = 0, \ldots, 4$. and hence $a = x_0 = -1$, $b = x_4 = 3$. We obtain

a) $f_1(x)$ is cubic polynomial on $[-1, 3]$ and hence a spline. (2 P.)

b) f_2 is not a polynomial of degree on the interval $[0, 1]$ since at $x = \frac{1}{2}$ the function is not differentiable. So both an argument that 1. or that 2. is not fulfilled is enough to conclude that this is not a spline. (2 P.)

c) We first resolve the definitions of $(\cdot)_+$ and obtain

$$f_3(x) = \begin{cases} 0 & \text{for } x \in [-1, 0), \\ 4x^2 & \text{for } x \in [0, 2), \\ 4x^2 + 2(x-2)^3 & \text{for } x \in [2, 3] \end{cases}$$

and we see that in every segment we have a polynomial of degree at most 3, so 1. is fulfilled

It remains to check the points 0, 2, 3 to see whether the function is continuous

We compute the first and second derivative

$$f_3'(x) = \begin{cases} 0 & \text{for } x \in [-1,0), \\ 8x & \text{for } x \in [0,2), \\ 8x + 6(x-2)^2 & \text{for } x \in [2,3]. \end{cases} \quad \text{and} \quad f_3''(x) = \begin{cases} 0 & \text{for } x \in [-1,0), \\ 8 & \text{for } x \in [0,2), \\ 8 + 12(x-2) & \text{for } x \in [2,3]. \end{cases}$$

The first derivative is still continuous, but the second is not, since

$$f_3''(0)\big|_{[-1,0]} = 0 \neq 8 = f_4''(0)\big|_{[0,1]}.$$

and we do not have a spline in this case. (3 P.)

d) The dimension of the space is $n + k$, where $n + 1$ is the number of nodes (since we denote them by $x_0, \ldots, x_n$), and k is the polynomial degree. Here we have $n + 1 = 5$ nodes so $n = 4$ and polynomial degree $k = 3$ so the dimension is 7.(1 P.)

e) A spline s is called *natural* if the second derivative vanishes at the boundary, here $s''(-1) = s''(3) = 0$. (2 P.)

Problem 3. (Fourier Series, 14 points)

a) Let the function $g(x) = -x$, $x \in [0, \pi]$, be given and consider the 2π-periodic function f_3 obtained from the *even extension* g_e of g defined on $[-\pi, \pi]$ by periodisation.
Sketch the function f_3 on at least 3 intervals.

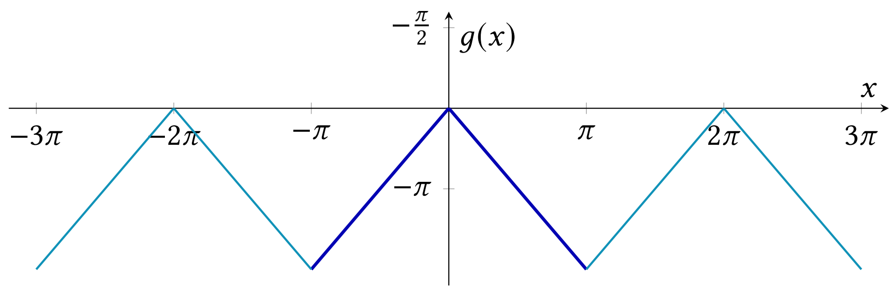
b) Compute all coefficients of the real Fourier series of f_3 from a).

c) Consider the 2π-periodic function $f_1(x) = 7e^{-ix} + 5e^{3ix} + 3e^{-5ix} + e^{7ix}$, $x \in [-\pi, \pi)$.
Compute all coefficients of the complex Fourier series of f_1.

d) Consider the 2π-periodic function $f_2(x) = 2(\cos(2x))^2 + 2\sin(4x) - 2\cos(8x)$, $x \in [-\pi, \pi)$.
Compute all coefficients of the complex Fourier series of f_2.

Solution.

a) The sketch looks like (3 P.)



b) We can use the the even extension is an even function. Hence $b_n = 0$ for $n = 1, 2, \ldots$.
For the a_n we use, that integrating over half an interval and multiplying that by 2 yields the result. Hence (1 P.)

$$a_0 = \frac{2}{2\pi}\int_0^\pi g(x)\mathrm{d}x = -\frac{1}{\pi}\int_0^\pi x\mathrm{d}x = -\frac{\pi}{2}.$$

and

$$a_n = \frac{2}{\pi}\int_0^\pi g(x)\cos(nx)\mathrm{d}x = -\frac{2}{\pi}\int_0^\pi x\cos(nx)\mathrm{d}x$$

where we apply integration by parts. We have in $\int fg'\mathrm{d}x = fg - \int f'g\mathrm{d}x$ here with $f(x) = x$ and $g'(x) = \cos(nx)$ so $f'(x) = 1$ and $g(x) = \frac{1}{n}\sin(nx)$ (2 P.)

$$-\frac{2}{\pi}\left(\frac{x}{n}\sin(nx)\Big|_0^\pi - \int_0^\pi 1\cdot\frac{1}{n}\sin(nx)\mathrm{d}x\right) = \frac{2}{n\pi}\int_0^\pi \sin(nx)\mathrm{d}x,$$

since $\sin(n\pi) = \sin(0) = 0$ for all n. Now we just have to integrate and obtain (1 P.)

$$a_n = -\frac{2}{\pi n^2}\cos(nx)\Big|_0^\pi = -\frac{2}{\pi n^2}\big(\cos(n\pi) - \cos(0)\big) = -\frac{2}{\pi n^2}\big((-1)^n - 1\big)$$

Since $\cos(0) = 1$ and $\cos(n\pi) = \begin{cases} 1 & \text{if } n \text{ even,} \\ -1 & \text{if } n \text{ odd,} \end{cases}$
the coefficient can further be simplified to

$$a_n = \begin{cases} \frac{4}{\pi n^2} & \text{if } n \text{ odd,} \\ 0 & \text{if } n \text{ even.} \end{cases}$$

c) We can directly read off $c_{-1} = 7$, $c_3(f_1) = 5$, $c_{-5}(f_1) = 3$, $c_7(f_1) = 1$ and all other coefficients are zero. (3 P.)

d) We can write $2(cos(2x))^2 = \cos(4x) + 1$. Hence the function reads

$$f_2(x) = \cos(4x) + 1 + 2\sin(4x) - 2\cos(8x)$$

and can directly read off the real coefficients $a_0 = 1$, $a_4 = 1$, $b_4 = 2$ and $a_8 = -2$ and all others are zero. This means for the complex coefficients, that (4 P.)

- $c_0 = a_0 = 1$
- $c_4 = \frac{1}{2}(a_4 - \mathrm{i}b_4) = \frac{1}{2} - \mathrm{i}$
- $c_{-4} = \frac{1}{2} + \mathrm{i}$ (either the same way as above or since f_2 is real)
- $c_8 = c_{-8} = -1$ (since $b_8 = 0$)
- all other coefficients are zero.

Problem 4. (Discrete Fourier Transform, 12 points)

In this task we consider the discrete Fourier Transform (DFT) for signals of length $n = 5$. We denote by $w_5 = \mathrm{e}^{-2\pi\mathrm{i}/5}$.

a) Using w_5, what does the Fourier matrix $\mathcal{F}_5$ look like?

b) Assume $\mathbf{f} = (f_0, f_1, f_2, f_3, f_4)$, is given by $f_i = g(x_j), x_j = \frac{2\pi j}{N}, i = 0, \ldots, 4$, i.e. sampling a 2π-periodic function g, which is known to be *bandlimited*, that is $c_k(g) = 0$ for $|k| > k_{\max} > 0$.
What is the highest frequency $k_{\max} \in \mathbb{N}$ where you can reconstruct g from $\hat{\mathbf{f}} = \mathcal{F}_5\mathbf{f}$?

What happens if frequencies $k > k_{\max}$ occur?

c) What does the inverse DFT $\mathcal{F}_5^{-1}\hat{\mathbf{f}}$ look like for $\hat{\mathbf{f}} = (c, 0, 0, 0, 0), c \in \mathbb{R}$?

Solution.

a) We have (4 P.)

$$\mathcal{F}_5 = \left(w_5^{jk}\right)_{j,k=0}^{4} = \begin{pmatrix} w_5^0 & w_5^0 & w_5^0 & w_5^0 & w_5^0 \\ w_5^0 & w_5^1 & w_5^2 & w_5^3 & w_5^4 \\ w_5^0 & w_5^2 & w_5^4 & w_5^1 & w_5^3 \\ w_5^0 & w_5^3 & w_5^1 & w_5^4 & w_5^2 \\ w_5^0 & w_5^4 & w_5^3 & w_5^2 & w_5^1 \end{pmatrix}$$

b) The highest frequency is $k_{\max} = 2$, (1 P.)
since periodicity of the discrete Fourier coefficients yields in this case (2 P.)

$$\hat{f}_0 = c_0(g), \quad \hat{f}_1 = c_1(g), \quad \hat{f}_2 = c_2(g), \quad \hat{f}_3 = c_{-2}(g), \quad \hat{f}_4 = c_{-1}(g).$$

If higher frequencies occur we observe the *aliasing effect* (1 P.)

c) This set of discrete Fourier coefficients stems from a bandlimited function g with only $\hat{f}_0 = Nc_0(g) = c$ and hence so we have $c_0(g) = \frac{c}{5}$ and $c_k(g) = 0, |k| > 0$ hence g is a constant function and using the same idea as in b) we get $\mathbf{f} = \frac{1}{5}(c, c, c, c, c)$.
(4 P.)

Problem 5. (Separation of Variables, 14 points)

Find all non-trivial solutions of the equation

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, \qquad \text{where } -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad \text{and } t > 0$$

that are of the form $u(x,t) = F(x)G(t)$ and that satisfy the boundary conditions

$$u(-\tfrac{\pi}{2}, t) = 0 \quad \text{and} \quad u(\tfrac{\pi}{2}, t) = 0 \qquad \text{for } t > 0.$$

What would you additionally need to obtain a unique solution?

Solution.

We insert the equation $u(x,t) = F(x)G(t)$ into the PDE and obtain the equation (1 P.)

$$F(x)\dot{G}(t) = 2F''(x)G(t).$$

Dividing by $2G(t)$ and $F(x)$ yields the equation (1 P.)

$$\frac{\dot{G}(t)}{2G(t)} = \frac{F''(x)}{F(x)} = k,$$

where k is some constant. From this we obtain the two ODEs (1 P.)

$$\left.\begin{aligned} F'' &= kF \\ \dot{G} &= 2kG \end{aligned}\right\}$$

We consider now the possible solutions of the equation for F. Thus we have three possibilities:

$k > 0$: Denote $p = \sqrt{k} > 0$. Then we have the solution

$$F(x) = Ae^{px} + Be^{-px}.$$

From the boundary conditions $F(\frac{\pi}{2}) = F(-\frac{\pi}{2}) = 0$ we get

$$F(\tfrac{\pi}{2}) = Ae^{p\frac{\pi}{2}} + Be^{-p\frac{\pi}{2}} = 0 \qquad \text{and } F(-\tfrac{\pi}{2}) = Ae^{-p\frac{\pi}{2}} + Be^{p\frac{\pi}{2}} = 0$$

Which is equivalent to $A = -Be^{-p\pi}$, plugging this into the second yields $B = 0$ and hence with the first $A = 0$. So this only yields the trivial solution, $F \sim 0$ we are not interested in. (2 P.)

$k = 0$: Here we have the ODE $F'' = 0$, which has the general solution

$$F(x) = A + Bx.$$

Now we get from the boundary conditions that

$$F(-\tfrac{\pi}{2}) = A - B\frac{\pi}{2} = 0,$$
$$F(\tfrac{\pi}{2}) = A + B\frac{\pi}{2} = 0.$$

Adding both yields $2A = 0$ and hence $A = 0$ subtracting both yields $\pi B = 0$ and hence $B = 0$.
We again only obtain the trivial solution. (2 P.)

$k < 0$: Denote $p = \sqrt{-k} > 0$. Then we have the solution

$$F(x) = A\cos(px) + B\sin(px).$$

Now the boundary conditions become (2 P.)

$$F(-\tfrac{\pi}{2}) = A\cos(-p\tfrac{\pi}{2}) + B\sin(-\tfrac{\pi}{2}p) = 0,$$
$$F(\tfrac{\pi}{2}) = A\cos(p\tfrac{\pi}{2}) + B\sin(p\tfrac{\pi}{2}) = 0.$$

Because the cosine is an even function and the sine is an odd function, we can rewrite this as

$$A\cos(p\tfrac{\pi}{2}) - B\sin(p\tfrac{\pi}{2}) = 0,$$
$$A\cos(p\tfrac{\pi}{2}) + B\sin(p\tfrac{\pi}{2}) = 0.$$

If we now add the two equations, we get that $A\cos(p\frac{\pi}{2}) = 0$. Thus either $A = 0$ or $\cos(p\frac{\pi}{2}) = 0$. Moreover, $\cos(p\frac{\pi}{2}) = 0$ if and only if $p = 2(n + 1/2)$ with $n = 1, 2, 3, \ldots$

If we subtract the second equation from the first, we get that $B\sin(p) = 0$, which holds if either $B = 0$ or $\sin(p\frac{\pi}{2}) = 0$, the latter implying that $p = 2n$ with $n = 1, 2, 3, \ldots$

We thus have two different types of solutions:

– On the one hand, we have the solutions (1 P.)

$$F(x) = \cos\big(2(n + 1/2)x\big) \qquad \text{for } n = 1, 2, \ldots$$

Here $p = 2(n+1/2)$ and $k = -p^2 = -4(n+1/2)^2$, and thus the corresponding solution for G is

$$G(t) = Ce^{-2kt} = Ce^{-8(n+1/2)^2 t}.$$

- On the other hand, we have the solutions (1 P.)

  $$F(x) = \sin(2nx) \qquad \text{for } n = 1, 2, \ldots$$

  Here $p = 2n$ and $k = -p^2 = -4n^2$, and thus the corresponding solution for G is

  $$G(t) = Ce^{2kt} = Ce^{-8n^2 t}.$$

In total, we have the non-trivial solutions (1 P.)

$$u(x,t) = Ae^{-8(n+1/2)^2 t} \cos\big(2(n+1/2)x\big) \qquad \text{for } n = 1, 2, \ldots$$

and

$$u(x,t) = Ae^{-8n^2 t} \sin\big(2nx\big). \qquad \text{for } n = 1, 2, \ldots$$

In order to obtain a unique solution we would need *initial conditions* $u(x,0) = f(x), x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. (2 P.)

Problem 6. (Laplace transform, 13 points)

Using the Laplace transform, solve the ordinary differential equation

$$y'' + 4y = (t-2)u(t-2)$$

with initial conditions $y(0) = y'(0) = 0$, and with u denoting the Heaviside function.

Solution.

Applying the Laplace transform to the ODE, we get (3 P.)

$$s^2Y(s) + 4Y(s) = \mathcal{L}((t-2)u(t-2)),$$

where $\mathcal{L}((t-2)u(t-2)) = \mathcal{L}(f(t-2)u(t-2)) = \mathrm{e}^{-2s}F(s)$ where we here have $f(t) = t$ and hence $F(s) = \frac{1}{s^2}$. (3 P.)

Plugging this in and dividing by $s^2 + 4$ we obtain (2 P.)

$$Y(s) = \mathrm{e}^{-2s}\frac{1}{s^2}\frac{1}{s^2+4}.$$

The first factor just yields a shift, so we are left to compute the inverse Laplace transform

$$\mathcal{L}^{-1}\Big(\frac{1}{s^2(s^2+4)}\Big)$$

which is a product Laplace domain, so a convolutions in time. It reads (4 P.)

$$\begin{aligned} t * \frac{1}{2}\sin(2t) &= \frac{1}{2}\int_0^t (t-\tau)\sin(2\tau)\mathrm{d}\tau \\ &= \frac{1}{2}\Big((t-\tau)(-\frac{1}{2}\cos(2\tau))\Big|_0^t - \int_0^t (-1)(-\frac{1}{2}\cos(2\tau))\mathrm{d}\tau\Big) \\ &= \frac{1}{2}\Big(0-\big(-t\cdot\frac{1}{2}\cdot 1t\big) - \frac{1}{4}\sin(2\tau)\Big|_0^t\Big) \\ &= \frac{1}{4}\big(t - \frac{1}{2}\sin(2t)\big). \end{aligned}$$

Alternatively. We can do a partial fraction decomposition of said term

$$\frac{1}{s^2(s^2+4)} = \frac{A}{s^2} + \frac{B}{s^2+4} \quad \Leftrightarrow 1 = 1 + 0s = A(s^2+4) + Bs^2 = (A+B)s^2 + 4A$$

Formally this would also include $A + Cs$ and $B + Ds$ but C, D lead to odd order terms that do not appear on the left.

By comparison of coefficients we obtain $A + B = 0$ so $A = -B$ and $1 = 4A$
Hence $A = \frac{1}{4}$ and $B = -\frac{1}{4}$ We can combine this to

$$\frac{1}{4}\left(\frac{1}{s^2} - \frac{1}{2} \cdot \frac{2}{s^2 + 2^2}\right)$$

which yields the same inverse Laplace transform as above, i.e. $y(t) = \frac{1}{4}\big(t - \frac{1}{2}\sin(2t)\big)$
so together with the shift we obtain (2 P.)

$$y(t) = \frac{1}{4}\big((t-2) - \frac{1}{2}\sin(2(t-2))\big)u(t-2)$$

Problem 7. (Convolution, 12 points)

Using the Laplace transform, solve the integro-differential equation

$$y(t) + 2\int_0^t y(\tau)e^{5(t-\tau)}\mathrm{d}\tau = \sin(3t) - \cos(3t).$$

Solution.

We compute the Laplace transform on both sides and obtain due to the conv9lution theorem that (3 P.)

$$Y(s) + \frac{2}{s-5}Y(s) = \frac{3}{s^2+9} - \frac{s}{s^2+9}$$

We simplify the right hand side (3 P.)

$$Y(s) + \frac{2}{s-5}Y(s) = \left(1 + \frac{2}{s-5}\right)Y(s) = \left(\frac{s-5+2}{s-5}\right)Y(s) = \frac{-(s-3)}{s^2+9}$$

Dividing by the factor in front of $Y(s)$ yields by splitting (3 P.)

$$Y(s) = \frac{-(s-5)}{s^2+9} = \frac{5}{3}\frac{3}{s^2+9} - \frac{s}{s^2+9}$$

and hence (3 P.)

$$y(t) = \frac{5}{3}\sin(3t) - \cos(3t)$$

Problem 8. (Numerical Methods for ODEs, 10 points)

We consider the following Runge–Kutta method for solving a scalar ODE, $y' = f(t, y)$, implemented in Python. The main block of the code looks like

```python
for n in range(N):
    yn = y[n]
    tn = t[n]
    k1 = f(tn, yn)
    k2 = f(tn+1/3*h,yn+1/3*h*k1)
    k3 = f(tn+2/3*h, yn+2/3*h*k2)
    y[n+1] = yn+h/4*(k1+3*k3)
```

Write down the Butcher-tableau, and determine the order and the number of stages of the method.

Solution.

The Butcher-tableau is given by (4 P.)

$$\begin{array}{c|ccc} 0 & & & \\ 1/3 & 1/3 & & \\ 2/3 & 0 & 2/3 & \\ \hline & 1/4 & 0 & 3/4 \end{array}$$

The method has $s = 3$ stages.

We have to check the order conditions. We use that $c_1 = 0$. (5 P.)

$$p = 1 \qquad b_1 + b_2 + b_3 = \frac{1}{4} + 0 + \frac{3}{4} = 1 \qquad \text{OK}$$

$$p = 2 \qquad b_1c_1 + b_2c_2 + b_3c_3 = \frac{1}{4} \cdot 0 + 0 \cdot \frac{1}{3} + \frac{3}{4} \cdot \frac{2}{3} = 1/2 \qquad \text{OK}$$

$$p = 3 \qquad b_1c_1^2 + b_2c_2^2 + b_3c_3^2 = \frac{1}{4} \cdot 0^2 + 0 \cdot \frac{1}{9} + \frac{3}{4} \cdot \frac{4}{9} = 1/3 \qquad \text{OK}$$

$$p = 3 \qquad b_3a_{32}c_2 = \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{1}{6} = 1/6 \qquad \text{OK}$$

$$p = 4 \qquad b_1c_1^3 + b_2c_2^3 + b_3c_3^3 = b_3c_3^3 = \frac{3}{4}\frac{2^3}{27} = \frac{2}{9} \neq \frac{1}{4}$$

The first order 4 condition is not satisfied, so the method is of order 3. (1 P.)

Formula Sheet.

Fourier Transform. The Fourier Transform $\hat{f} = \mathcal{F}(f)$ and its inverse $f = \mathcal{F}^{-1}(\hat{f})$ are

$$\hat{f}(\omega) = \mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-\mathrm{i}\omega x}\, \mathrm{d}x \quad \text{and} \quad f(x) = \mathcal{F}^{-1}(\hat{f})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) \mathrm{e}^{\mathrm{i}\omega x}\, \mathrm{d}\omega$$

Laplace Transform. The Laplace transform $F(s)$ of $f(t)$, $t \geq 0$, reads

$$F(s) = \int_0^{\infty} \mathrm{e}^{-st} f(t)\, \mathrm{d}t$$

List of Fourier Transforms.

$f(x)$	$\hat{f}(\omega)$
e^{-ax^2}	$\frac{1}{\sqrt{2a}} \mathrm{e}^{-\frac{\omega^2}{4a}}$
$\mathrm{e}^{-a\lvert x\rvert}$	$\sqrt{\frac{2}{\pi}} \frac{a}{\omega^2 + a^2}$
$\frac{1}{x^2 + a^2}$ for $a > 0$	$\sqrt{\frac{\pi}{2}} \frac{\mathrm{e}^{-a\lvert\omega\rvert}}{a}$
$\begin{cases} 1 & \text{for } \lvert x\rvert < a \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(\omega a)}{\omega}$
$\sqrt{\frac{2}{\pi}} \frac{\sin(ax)}{x}$	$\begin{cases} 1 & \text{for } \lvert\omega\rvert < a \\ 0 & \text{otherwise} \end{cases}$

List of Laplace Transforms.

$f(t)$	$F(s)$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2 - \omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2 - \omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s - a}$
$f(t-a)u(t-a)$	$\mathrm{e}^{-sa} F(s)$
$\delta(t-a)$	e^{-sa}

Trigonometric identities.

- $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$
- $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$
- $\cos(2\alpha) = 2\cos^2(\alpha) - 1 = 1 - 2\sin^2(\alpha)$
- $\mathrm{e}^{\mathrm{i}\alpha} = \cos\alpha + \mathrm{i}\sin\alpha$
- $2\sin\alpha \cos\beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)$
- $2\cos\alpha \sin\beta = \sin(\alpha + \beta) - \sin(\alpha - \beta)$
- $2\cos\alpha \cos\beta = \cos(\alpha - \beta) + \cos(\alpha + \beta)$
- $2\sin\alpha \sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$

Integrals.

- $\int x^n \cos ax \mathrm{d}x = \frac{1}{a} x^n \sin ax - \frac{n}{a} \int x^{n-1} \sin ax \mathrm{d}x$
- $\int x^n \sin ax \mathrm{d}x = -\frac{1}{a} x^n \cos ax + \frac{n}{a} \int x^{n-1} \cos ax \mathrm{d}x$

Order conditions for Runge-Kutta methods

p	Conditions	p	Conditions
1	$\sum_{i=1}^{s} b_i = 1$	4	$\sum_{i=1}^{s} b_i {c_i}^3 = \frac{1}{4}$
2	$\sum_{i=1}^{s} b_i c_i = \frac{1}{2}$		$\sum_{i=1}^{s} \sum_{j=1}^{s} b_i c_i a_{ij} c_j = \frac{1}{8}$
3	$\sum_{i=1}^{s} b_i {c_i}^2 = \frac{1}{3}$		$\sum_{i=1}^{s} \sum_{j=1}^{s} b_i a_{ij} {c_j}^2 = \frac{1}{12}$
	$\sum_{i=1}^{s} \sum_{j=1}^{s} b_i a_{ij} c_j = \frac{1}{6}$		$\sum_{i=1}^{s} \sum_{j=1}^{s} \sum_{k=1}^{s} b_i a_{ij} a_{jk} c_k = \frac{1}{24}$