③ (Inverse Laplace)

$$4(\mathrm{I}) = \int_0^t e^{i(t-y)}\left(e^{iy}+e^{-iy}\right)dy$$

$$= e^{it}\int_0^t e^{-iy}\left(e^{iy}+e^{-iy}\right)dy$$

$$= e^{it}\int_0^t \left(1+e^{-2iy}\right)dy$$

$$= e^{it}\left(t + \left[\frac{e^{-2iy}}{-2i}\right]_0^t\right)$$

$$= e^{it}\left(t + \frac{1}{2i} - \frac{1}{2i}e^{-2it}\right).$$

$$4(\mathrm{II}) = \int_0^t e^{-i(t-y)}\left(e^{iy}+e^{-iy}\right)dy$$

$$= e^{-it}\int_0^t e^{iy}\left(e^{iy}+e^{-iy}\right)dy$$

$$= e^{-it}\int_0^t e^{2iy}+1\,dy$$

$$= e^{-it}\left(t + \left[\frac{e^{2iy}}{2i}\right]_0^t\right)$$

$$= e^{-it}\left(t + \frac{e^{2it}}{2i} - \frac{1}{2i}\right)$$

$$4(\mathrm{I}) + 4(\mathrm{II}) = e^{it}\left(\cancel{t} + \cancel{\frac{1}{2i}} - \frac{e^{-2it}}{2i}\right)$$

$$+ e^{-it}\left(\cancel{t} + \frac{e^{2it}}{2i} - \cancel{\frac{1}{2i}}\right)$$

$$= t\left(e^{it} + e^{-it}\right) + \frac{1}{2i}\left(e^{it} - e^{-it}\right)$$

$$+ \frac{1}{2i}\left(e^{it} - e^{-it}\right)$$

$$= 2t\cos(t) + 2\sin(t).$$

$$\left(\cos(t) = \frac{1}{2}\left(e^{it} + e^{-it}\right), \quad \sin(t) = \frac{1}{2i}\left(e^{it} - e^{-it}\right)\right)$$

$$\cos(t) * \cos(t) = (\mathrm{I}) + (\mathrm{II})$$

$$= \underline{\frac{1}{2}\left(t\cos(t) + \sin(t)\right)}$$

$$\underline{y(t) = \frac{1}{2}\left(t\cos(t) + \sin(t)\right).}$$

① (Laplace ODE)

Ønsker å regne ut $\sin(t) * \sin(t)$.

$$\sin(t) * \sin(t) = \int_0^t \sin(t-y)\sin(y)\,dy \qquad (*)$$

$$\sin(t) = \frac{1}{2i}\left(e^{it} - e^{-it}\right)$$

$$(*) \left(\frac{1}{2i}\right)^2 \int_0^t \left(e^{i(t-y)} - e^{-i(t-y)}\right)\left(e^{iy} - e^{-iy}\right)dy$$

$$= -\frac{1}{4}\int_0^t \underbrace{e^{i(t-y)}\left(e^{iy} - e^{-iy}\right)}_{(I)} - \underbrace{e^{-i(t-y)}\left(e^{iy} - e^{-iy}\right)}_{(II)}dy$$

$$(I) = \int_0^t e^{i(t-y)}\left(e^{iy} - e^{-iy}\right)dy$$

$$= e^{it}\int_0^t e^{-iy}\left(e^{iy} - e^{-iy}\right)dy$$

$$= e^{it}\int_0^t \left(1 - e^{-2iy}\right)dy$$

$$= e^{it}\left(t - \left[\frac{e^{-2iy}}{-2i}\right]_0^t\right)$$

$$= e^{it}\left(t + \frac{e^{-2it}}{2i} - \frac{1}{2i}\right)$$

$$(II) = \int_0^t e^{-i(t-y)}\left(e^{iy} - e^{-iy}\right) dy$$

$$= e^{-it}\int_0^t e^{iy}\left(e^{iy} - e^{-iy}\right) dy$$

$$= e^{-it}\int_0^t \left(e^{2iy} - 1\right) dy$$

$$= e^{-it}\left(-t + \left[\frac{e^{2iy}}{2i}\right]_0^t\right)$$

$$= e^{-it}\left(-t + \frac{e^{2it}}{2i} - \frac{1}{2i}\right).$$

$$(I) - (II) = e^{it}\left(\cancel{t} + \frac{e^{-2it}}{\cancel{2i}} - \frac{1}{\cancel{2i}}\right)$$

$$- e^{-it}\left(-\cancel{t} + \frac{e^{2it}}{\cancel{2i}} - \frac{1}{\cancel{2i}}\right)$$

$$= 2t\cos(t) - \sin(t) - \sin(t)$$

$$= 2t\cos(t) - 2\sin(t)$$

$$\cos(t) = \frac{1}{2}\left(e^{it} + e^{-it}\right), \quad -\frac{e^{it}}{2i} + \frac{e^{-it}}{2i} = -\frac{1}{2i}\left(e^{it} - e^{-it}\right)$$

$$\sin(t) * \sin(t) = \frac{1}{2}(\sin(t) - t\cos(t))$$

$$Y(s) = \mathcal{L}(1) \cdot \mathcal{L}(\sin(t) * \sin(t))$$

$$= \mathcal{L}(1)\,\mathcal{L}\left(\frac{1}{2}(\sin(t) - t\cos(t))\right)$$

$$= \mathcal{L}\left(1 * \frac{1}{2}(\sin(t) - t\cos(t))\right)$$

$$= \mathcal{L}\left(\int_0^t \frac{1}{2}(\sin(y) - y\cos(y))\,dy\right) \qquad (*)$$

$$\frac{1}{2}\int_0^t \sin(y)\,dy = \frac{1}{2}\left[-\cos(y)\right]_0^t = \frac{1}{2}(1-\cos(t))$$

$$-\frac{1}{2}\int_0^t y\cos(y)\,dy = -\frac{1}{2}\left[y\sin(y) + \cos(y)\right]_0^t$$

$$= -\frac{1}{2}(t\sin(t) + \cos(t) - 1)$$

$$(*) = \mathcal{L}\left(\frac{1}{\cancel{2}}(1-\cos(t)) - \frac{1}{2}t\sin(t) - \frac{\cos(t)}{2} + \frac{1}{\cancel{2}}\right)$$

$$= \mathcal{L}\left(1 - \cos(t) - \frac{1}{2}t\sin(t)\right)$$

$$\underline{y(t) = 1 - \cos(t) - \frac{t}{2}\sin(t)}$$

$$(y\sin(y) + \cos(y))'$$

$$= \sin(y) + y\cos(y) - \sin(y) = y\cos(y)$$