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a)

$$f(t) = \begin{cases} e^t, & x \in [a, b] \\ 0, & \text{ellers} \end{cases}$$

$$\mathcal{F}(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cdot e^{-i\omega t} \, dt$$

$$\sqrt{2\pi}\, \mathcal{F}(f) = \int_a^b e^t \cdot e^{-i\omega t} \, dt$$

$$= \int_a^b e^{(1-i\omega)t} \, dt$$

$$= \left[\frac{e^{(1-i\omega)t}}{1-i\omega} \right]_a^b$$

$$= \frac{e^{(1-i\omega)b} - e^{(1-i\omega)a}}{1-i\omega}$$

b) $f(t) = e^{-|t|} \sin(t)$

$$\sqrt{2\pi}\, \mathcal{F}(f) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} \, dt$$

$$\sin(t) = \frac{1}{2}\left(e^{it} - e^{-it}\right)$$

$$= \int_{-\infty}^{\infty} e^{-|t|} \frac{1}{2}\left(e^{it} - e^{-it}\right) e^{-i\omega t}\, dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \Big(\underbrace{e^{-|t| + it - i\omega t}}_{(\mathrm{I})} - \underbrace{e^{-|t| - it - i\omega t}}_{(\mathrm{II})} \Big)\, dt$$

$$(\mathrm{I}) = \int_{-\infty}^{0} e^{t + it - i\omega t}\, dt + \int_{0}^{\infty} e^{-t + it - i\omega t}\, dt$$

$$= \left[\frac{e^{(1 + i - i\omega) t}}{1 + i - i\omega} \right]_{-\infty}^{0}$$

$$+ \left[\frac{e^{(-1 + i - i\omega) t}}{-1 + i - i\omega} \right]_{0}^{\infty}$$

$$= \frac{1}{1 + i - i\omega} - 0$$

$$+ \; 0 - \frac{1}{-1 + i - i\omega}$$

$$= \frac{1}{1 + (1-\omega)i} + \frac{1}{1 - (1-\omega)i}$$

$$= \frac{2}{(1+(1-\omega)i)(1-(1-\omega)i)} = \frac{2}{1+(1-\omega)^2}$$

$$= (I).$$

$$(II) = \int_{-\infty}^{\infty} e^{-|t|-it-i\omega t}\,dt$$

$$= \int_{-\infty}^{0} e^{t-it-i\omega t}\,dt + \int_{0}^{\infty} e^{-t-it-i\omega t}\,dt$$

$$= \left[\frac{e^{(1-i-i\omega)t}}{1-i-i\omega}\right]_{-\infty}^{0} + \left[\frac{e^{(-1-i-i\omega)t}}{-1-i-i\omega}\right]_{0}^{\infty}$$

$$= \frac{1}{1-i-i\omega} + \frac{-1}{-1-i-i\omega}$$

$$= \frac{1}{1-(1+\omega)i} + \frac{1}{1+(1+\omega)i}$$

$$= \frac{1+(1+\omega)i+1-(1+\omega)i}{1+(1+\omega)^2} = \frac{2}{1+(1+\omega)^2} = (II)$$

$$\sqrt{2\pi}\, \mathcal{F}(f) = \frac{1}{2}\left((\mathrm{I}) - (\mathrm{II}) \right)$$

$$= \frac{1}{2}\left(\frac{2}{1+(1-\omega)^2} - \frac{2}{1+(1+\omega)^2} \right)$$

$$= \frac{1}{1+(1-\omega)^2} - \frac{1}{1+(1+\omega)^2} .$$

c)

$$f(t) = e^{-|t|} \cos(t), \qquad \cos(t) = \frac{1}{2}\left(e^{it} + e^{-it}\right)$$

$$= \frac{1}{2} e^{-|t|} \left(e^{it} + e^{-it} \right)$$

$$\sqrt{2\pi}\, \mathcal{F}(f) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{2} e^{-|t|} \left(e^{it} + e^{-it}\right) e^{-i\omega t}\, dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \underbrace{e^{-|t| + it - i\omega t}}_{(\mathrm{I})} + \underbrace{e^{-|t| - it - i\omega t}}_{(\mathrm{II})}\, dt$$

$$(\mathrm{I}) = \int_{-\infty}^{0} e^{t + it - i\omega t}\, dt + \int_{0}^{\infty} e^{-t + it - i\omega t}\, dt$$

$$= \left[\frac{e^{(1+i-i\omega)t}}{1+i-i\omega} \right]_{-\infty}^{0} + \left[\frac{e^{(-1+i-i\omega)t}}{-1+i-i\omega} \right]_{0}^{\infty}$$

$$= \frac{1}{1+i-i\omega} + 0 - \frac{1}{-1+i-i\omega}$$

$$= \frac{1}{1+(1-\omega)i} + \frac{1}{1-(1-\omega)i} \qquad (*)$$

$$= \frac{2}{1+(1-\omega)^2} = (I)$$

$$(*) = \frac{1-(1-\omega)i}{(1+(1-\omega)i)(1-(1-\omega)i)} + \frac{1+(1-\omega)i}{(1-(1-\omega)i)(1+(1-\omega)i)}$$

$$= \frac{2}{1+(1-\omega)^2}$$

$$(\mathrm{II}) = \int_{-\infty}^{\infty} e^{-|t| - it - i\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{t - it - i\omega t}\, dt + \int_{0}^{\infty} e^{-t - it - i\omega t}\, dt$$

$$= \left[\frac{e^{(1-i-i\omega)t}}{1-i-i\omega} \right]_{-\infty}^{0} + \left[\frac{e^{(-1-i-i\omega)t}}{-1-i-i\omega} \right]_{0}^{\infty}$$

$$= \frac{1}{1-i-i\omega} + 0 - \frac{1}{-1-i-i\omega}$$

$$= \frac{1}{1-(1+\omega)i} + \frac{1}{1+(1+\omega)i}$$

$$= \frac{2}{1+(1+\omega)^2} = (\mathrm{II})$$

$$\mathcal{F}(f) = \frac{1}{\sqrt{2\pi}} \frac{1}{2} \left((\mathrm{I}) + (\mathrm{II}) \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1+(1-\omega)^2} + \frac{1}{1+(1+\omega)^2} \right).$$