Plenumsregning matte 4N
23.01

Problem 1 (Taylor expansion)

Compute the Taylor expansion of $f(x) = (1-x)^{-1}$ about 0, (at some $|x| < 1$).

$f'(x) = (1-x)^{-2}$

$f''(x) = 2(1-x)^{-3}$

$f^{(3)}(x) = 3 \cdot 2 \cdot (1-x)^{-4}$

$\vdots$

$f^{(k)}(x) = k!\,(1-x)^{-(k+1)}$

$$T^m f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} x^k$$

$$T^m f(x) = \sum_{k=0}^{m-1} x^k$$

Problem 2 (Taylor expansion)

Show that $|(1-x)^{-1} - e^x| \leq Mx^2$ for some $M > 0$ (where $|x| < 1$).

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \qquad (1-x)^{-1} = 1 + x + x^2 + \cdots$$

$$(1-x)^{-1} - e^x = 0 + 0 + x^2 - \frac{1}{2}x^2 + O(x^3)$$

$$= \frac{1}{2}x^2 + O(x^3) = O(x^2). \quad (|x| < 1)$$

Comment:

(Note that M depends on x: we can take for example $M = \frac{1}{1-x} + \frac{1}{2}$. If we only considered $|x| < \delta < 1$, then M could be chosen independently of x, take for example $\frac{1}{1-\delta} + \frac{1}{2}$).

Problem 3 (numerical differentiation)

Show that $\frac{u(t+h)-u(t-h)}{2h} = u'(x) + \mathcal{O}(h^p)$, and find p.

$$T^n u(t+h) = \sum_{k=0}^{n-1} \frac{u^{(k)}(t)}{k!} h^k$$

$$T^n u(t-h) = \sum_{k=0}^{n-1} \frac{u^{(k)}(t)}{k!} (-h)^k$$

Ser på telleren ; $\frac{u(t+h)-u(t-h)}{2h}$:

$k=0$: $u(t) - u(t) = 0$

$k=1$: $u'(t)h + u'(t)h = 2h\,u'(t)$

$k=2$: $\frac{u''(t)}{2}h^2 - \frac{u''(t)}{2}h^2 = 0$

$k=3$: $\frac{2}{3!}u^{(3)}(t)h^3 \neq 0$.

$$\frac{u(t+h)-u(t-h)}{2h}$$

$$= \frac{0 + 2h\,u'(t) + 0 + \frac{2}{3!}u^{(3)}(t)h^3 + O(h^4)}{2h}$$

$$= u'(t) + O(h^2) \qquad (p=2).$$

Problem 4 (numerical differentiation)

Find a_1, a_2 such that $\frac{a_1 u(t+h) - a_2 u(t-h/2)}{h} = u'(t) + \mathcal{O}(h^1)$.

$$T^n u(t+h) = \sum_{k=0}^{n-1} \frac{u^{(k)}(t)}{k!} h^k$$

$$T^n u(t-\tfrac{h}{2}) = \sum_{k=0}^{n-1} \frac{u^{(k)}(t)}{k!} \left(-\frac{h}{2}\right)^k$$

sammenligner ledd

$k=0$: $a_1 u(t) - a_2 u(t) \Rightarrow \underline{a_1 = a_2 = a}$

$k=1$: $a\, u'(t) h + \frac{h}{2} a\, u'(t)$

$= u'(t)\, a\, h \frac{3}{2} \Rightarrow \underline{a = \frac{2}{3}}.$

$k=2$: $a \frac{u''(t)}{2} h^2 - a \frac{u''(t)}{2} \frac{h^2}{4} \neq 0$

$\underline{a_1 = a_2 = 2/3}$

Problem 5 (numerical differentiation)

Show that $\frac{u(t+h)-2u(t)+u(t-h)}{h^2} = u''(t) + \mathcal{O}(h^2)$.

$$T^m u(t+h) = \sum_{k=0}^{m-1} \frac{u^{(k)}(t)}{k!} h^k,$$

$$T^m u(t-h) = \sum_{k=0}^{m-1} \frac{u^{(k)}(t)}{k!} (-h)^k$$

Sammenligner ledd:

$k=0$: $u(t) - 2u(t) + u(t) = 0$

$k=1$: $u'(t)h - u'(t)h = 0$

$k=2$: $\frac{u''(t)h}{2} h^2 + \frac{u''(t)}{2} h^2 = \underline{u''(t)h^2}$

$k=3$: $\frac{u^{(3)}(t)h^3}{3!} - \frac{u^{(3)}(t)h^3}{3!} = 0$

$k=4$: $\frac{u^{(4)}(t)h^4}{4!} + \frac{u^{(4)}(t)h^4}{4!} \neq 0$

$$\frac{u(t+h) - 2u(t) + u(t-h)}{h^2}$$

$$= \frac{0 + 0 + u''(t)h^2 + \mathcal{O}(h^4)}{h^2}$$

$$= \underline{u''(t) + \mathcal{O}(h^2)}$$

Problem 6 (numerical solution of boundary value problem)

1. Show that $u(x) = -\sin(2\pi x)/(4\pi^2)$ is a solution to the boundary value problem $\partial_x^2 u = sin(2\pi x)$, $x \in (0,1), u(0) = u(1) = 0$.
2. Set up a finite difference scheme approximation for this BVP using central differences. Use a uniform grid with step size $h = 1/N$, so that $x_k = kh, k = 0, \ldots, N$.

(1) $u(0) = \sin(0) = u(1) = \sin(2\pi) = 0$ ok!

$$\partial_x u(x) = \partial_x \left(-\frac{1}{4\pi^2}\sin(2\pi x)\right) = -\frac{1}{2\pi}\cos(2\pi x)$$

$$\partial_x^2 u(x) = \partial_x \left(-\frac{1}{2\pi}\cos(2\pi x)\right) = \sin(2\pi x)$$ ok!