Problem 1.

Using the Laplace transform, solve the ordinary differential equation

$$2y'' + y' - y = 3u(t-2)\,,$$

with initial conditions $y(0) = y'(0) = 0$, and with u denoting the Heaviside function.

Solution.

Applying the Laplace transform to the ODE, we get

$$2[s^2Y(s)] + sY(s) - Y(s) = 3\frac{e^{-2s}}{s}\,,$$

which gives us

$$Y(s) = \frac{3e^{-2s}}{s(2s^2+s-1)} = e^{-2s}F(s)\,,$$

where

$$F(s) = \frac{3}{s(2s^2+s-1)}$$

is the Laplace transform of a function $f(t)$ we need to find. To use the second shift theorem, we first determine $f(t)$, then write $y(t) = u(t-2)f(t-2)$. Decomposition of $F(s)$ into partial fractions is done via

$$F(s) = \frac{3}{s(2s-1)(s+1)} = \frac{A}{s} + \frac{B}{2s-1} + \frac{C}{s+1}\,.$$

To find the coefficients, we have to satisfy

$$3 = A(2s-1)(s+1) + Bs(s+1) + Cs(2s-1) \;\text{ for all }\; s\,.$$

In particular, setting $s = 0$ gives immediately $A = -3$, while using $s = 1/2$ results in $B = 4$, so that $C = 1$ (alternatively, one can solve a 3×3 linear system to find A, B, C, which of course will give the same values). Hence:

$$F(s) = \frac{-3}{s} + \frac{4}{2s-1} + \frac{1}{s+1} = \frac{-3}{s} + \frac{2}{s-1/2} + \frac{1}{s+1} \quad\Rightarrow\quad f(t) = -3 + 2e^{t/2} + e^{-t}.$$

Finally, we apply the shift: $y(t) = u(t-2)f(t-2) = u(t-2)(-3 + 2e^{-1+t/2} + e^{2-t})$.

Problem 2.

Using the Laplace transform, solve the integro-differential equation

$$y'(t) + 5\int_0^t e^{2\tau} y(t-\tau)\,d\tau = e^{2t},$$

with the initial condition $y(0) = 0$.

Solution.

This equation can be written as $y'(t) + 5y(t) * f(t) = f(t)$, where $f(t) = e^{2t}$. Applying the Laplace transform to both sides, we get

$$[sY(s) - y(0)] + 5Y(s)F(s) = F(s)$$

and, from the table of Laplace transforms, we have $F(s) = (s-2)^{-1}$. Hence:

$$\left[s + \frac{5}{s-2}\right]Y(s) = \frac{1}{s-2} \quad\Rightarrow\quad \left[\frac{s^2 - 2s + 5}{s-2}\right]Y(s) = \frac{1}{s-2} \quad\Rightarrow\quad (s^2 - 2s + 5)Y(s) = 1,$$

that is,

$$Y(s) = \frac{1}{s^2 - 2s + 5}.$$

To find the inverse transform $y(t)$, there are (at least) two possibilities. The quickest one is probably by completing the square:

$$Y(s) = \frac{1}{s^2 - 2s + 1 + 4} = \frac{1}{(s-1)^2 + 2^2} = \frac{1}{2}\left[\frac{2}{(s-1)^2 + 2^2}\right].$$

Looking up on the table of Laplace transforms, we see that

$$\mathcal{L}(\sin 2t) = \frac{2}{s^2 + 2^2},$$

of which our $Y(s)$ is a *shifted* version. Using then the first shift theorem, we finally get

$$y(t) = \frac{1}{2}e^t \sin 2t,$$

where the exponential factor e^t appears due to the $s-1$ shift on $Y(s)$.

Alternatively: It is also possible to use partial fractions, considering complex numbers. The denominator $s^2 - 2s + 5$ has two roots:

$$s_\pm = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 2 \cdot 5}}{2} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i\,.$$

With that, we can write

$$Y(s) = \frac{1}{[s-(1-2i)][s-(1+2i)]} = \frac{A}{s-1+2i} + \frac{B}{s-1-2i}\,.$$

Solving for A and B, we get $A = i/4$ and $B = -i/4$. Then:

$$Y(s) = \frac{i}{4} \cdot \frac{1}{s-(1-2i)} - \frac{i}{4} \cdot \frac{1}{s-(1+2i)} = \frac{i}{4}\left[\frac{1}{s-(1-2i)} - \frac{1}{s-(1+2i)}\right],$$

so using the table of transforms we find

$$y(t) = \frac{i}{4}\left[\mathrm{e}^{(1-2i)t} - \mathrm{e}^{(1+2i)t}\right] = \frac{i}{4}\left[\mathrm{e}^t\mathrm{e}^{-2it} - \mathrm{e}^t\mathrm{e}^{2it}\right] = \frac{i\mathrm{e}^t}{4}\left[\mathrm{e}^{(-2t)i} - \mathrm{e}^{(2t)i}\right]\,.$$

Using Euler's identity, $\mathrm{e}^{\theta i} = \cos\theta + i\sin\theta$, we can rewrite $y(t)$ as

$$y(t) = \frac{i\mathrm{e}^t}{4}\left[\cos(-2t) + i\sin(-2t) - (\cos 2t + i\sin 2t)\right] = \frac{i\mathrm{e}^t}{4}\left[-2i\sin 2t\right] = \frac{\mathrm{e}^t\sin 2t}{2}\,.$$

Problem 3.

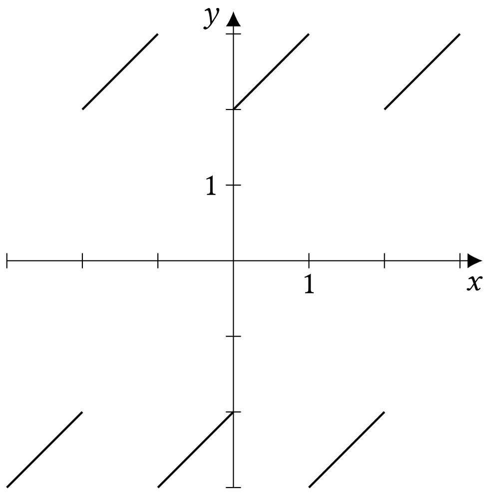
The function $f\colon [0,1] \to \mathbb{R}$ given by

$$f(x) = 2 + x \qquad \text{for } 0 \le x \le 1$$

is to be extended to an odd function g with period 2. Sketch the graph of g on the interval $[-3,3]$ and compute the Fourier series of g.

Solution.

The odd extension g of f is shown in the picture below:



Since g is an odd function, all its Fourier cosine coefficients a_n (including a_0) are equal to 0 and we have

$$\begin{aligned}
b_n &= 2\int_0^1 (2+x)\sin(n\pi x)\,dx \\
&= 4\int_0^1 \sin(n\pi x)\,dx + 2\int_0^1 x\sin(n\pi x)\,dx \\
&= -4\frac{\cos(n\pi x)}{n\pi}\bigg|_0^1 - \frac{2x\cos(n\pi x)}{n\pi}\bigg|_0^1 + \frac{2}{n\pi}\int_0^1 \cos(n\pi x)\,dx \\
&= -4\frac{\cos(n\pi)}{n\pi} + 4\frac{1}{n\pi} - 2\frac{\cos(n\pi)}{n\pi} + 0 + \frac{2\sin(n\pi x)}{n^2\pi^2}\bigg|_0^1 \\
&= \frac{2}{n\pi} - 6\frac{(-1)^{n+1}}{n\pi}.
\end{aligned}$$

Thus we get the Fourier series

$$g(x) = \sum_{n=1}^{\infty} \frac{2+6(-1)^n}{n\pi}\sin(n\pi x).$$

Problem 4.

Find all non-trivial solutions of the equation

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} \qquad \text{where } -1 < x < 1 \quad \text{and } t > 0$$

that are of the form $u(x,t) = F(x)G(t)$ and that satisfy the boundary conditions

$$u(-1,t) = 0 \qquad \text{and} \qquad u(1,t) = 0 \qquad \text{for } t > 0.$$

Solution.

We insert the equation $u(x,t) = F(x)G(t)$ into the PDE and obtain the equation

$$F(x)\dot{G}(t) = 4F''(x)G(t).$$

Dividing by $4G(t)$ and $F(x)$ yields the equation

$$\frac{\dot{G}(t)}{4G(t)} = \frac{F''(x)}{F(x)} = k,$$

where k is some constant. From this we obtain the two ODEs

$$\left.\begin{aligned} F'' &= kF \\ \dot{G} &= 4kG \end{aligned}\right\}$$

We consider now the possible solutions of the equation for F. Thus we have three possibilities:

$k > 0$: Denote $p = \sqrt{k} > 0$. Then we have the solution

$$F(x) = Ae^{px} + Be^{-px}.$$

From the boundary conditions we now get that

$$\begin{aligned} F(-1) &= Ae^{-p} + Be^{p} = 0, \\ F(1) &= Ae^{p} + Be^{-p} = 0. \end{aligned}$$

From the first equation we get that $A = -Be^{2p}$. Inserting that in the second equation, we then obtain that

$$0 = -Be^{3p} + Be^{-p} = -B(e^{3p} - e^{-p}).$$

This is only possible for $B = 0$. From $A = -Be^{2p}$, we then get that A needs to be 0 as well. Thus we only end up with the trivial solution.

$k = 0$: Here we have the ODE $F'' = 0$, which has the general solution

$$F(x) = A + Bx.$$

Now we get from the boundary conditions that

$$F(-1) = A - B = 0,$$
$$F(1) = A + B = 0.$$

By adding these equations, we get that $A = 0$; by subtracting the second equation from the first, we get that $B = 0$. Thus we obtain, again, only trivial solutions.

$k < 0$: Denote $p = \sqrt{-k} > 0$. Then we have the solution

$$F(x) = A\cos(px) + B\sin(px).$$

Now the boundary conditions become

$$F(-1) = A\cos(-p) + B\sin(-p) = 0,$$
$$F(1) = A\cos(p) + B\sin(p) = 0.$$

Because the cosine is an even function and the sine is an odd function, we can rewrite this as

$$A\cos(p) - B\sin(p) = 0,$$
$$A\cos(p) + B\sin(p) = 0.$$

If we now add the two equations, we get that $A\cos(p) = 0$. Thus either $A = 0$ or $\cos(p) = 0$. Moreover, $\cos(p) = 0$ if and only if $p = (n + 1/2)\pi$ with $n = 1, 2, 3, \ldots$

If we subtract the second equation from the first, we get that $B\sin(p) = 0$, which holds if either $B = 0$ or $\sin(p) = 0$, the latter implying that $p = n\pi$ with $n = 1, 2, 3, \ldots$

We thus have two different types of solutions:

- On the one hand, we have the solutions

$$F(x) = \cos\big((n + 1/2)\pi x\big) \qquad \text{for } n = 1, 2, \ldots$$

  Here $p = (n + 1/2)\pi$ and $k = -p^2 = -(n + 1/2)^2\pi^2$, and thus the corresponding solution for G is

$$G(t) = Ce^{-4kt} = Ce^{-4(n+1/2)^2\pi^2 t}.$$

- On the other hand, we have the solutions

$$F(x) = \sin(n\pi x) \qquad \text{for } n = 1, 2, \ldots$$

  Here $p = n\pi$ and $k = -p^2 = -n^2\pi^2$, and thus the corresponding solution for G is

$$G(t) = Ce^{-4kt} = Ce^{-4n^2\pi^2 t}.$$

In total, we have the non-trivial solutions

$$u(x,t) = Ae^{-4(n+1/2)^2\pi^2 t}\cos\big((n+1/2)\pi x\big) \qquad \text{for } n = 1, 2, \ldots$$

and

$$u(x,t) = Ae^{-4n^2\pi^2 t}\sin\big(n\pi x\big). \qquad \text{for } n = 1, 2, \ldots$$

Problem 5.

Use the Fourier transform in order to solve the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + u \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0$$

with initial conditions

$$u(x,0) = \begin{cases} \dfrac{\sin(x)}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Solution.

By taking the Fourier transform of the equation with respect to the x-variable we obtain the equation

$$\frac{\partial \hat{u}}{\partial t}(\omega,t) = i\omega\hat{u}(\omega,t) + \hat{u}(\omega,t),$$

which simplifies to

$$\frac{\partial \hat{u}}{\partial t}(\omega,t) = (1+i\omega)\hat{u}(\omega,t).$$

For fixed ω, the solution of this equation is

$$\hat{u}(\omega,t) = C(\omega)e^{(1+i\omega)t}$$

(where the “constant” C depends on the frequency ω). In particular, we obtain for $t = 0$ the equation

$$\hat{u}(\omega,0) = C(\omega).$$

On the other hand, we can compute the Fourier transform of the initial condition, which gives

$$C(\omega) = \hat{u}(\omega,0) = \mathcal{F}\Big(\frac{\sin x}{x}\Big)(\omega) = \begin{cases} \sqrt{\dfrac{\pi}{2}} & \text{for } |\omega| < 1, \\ 0 & \text{for } |\omega| > 1. \end{cases}$$

We obtain therefore that

$$\hat{u}(\omega, t) = C(\omega) e^{(1+i\omega)t} = \begin{cases} \sqrt{\frac{\pi}{2}} e^{(1+i\omega)t} & \text{for } |\omega| < 1, \\ 0 & \text{for } |\omega| > 1. \end{cases}$$

Finally, we obtain the actual solution $u(x, t)$, by applying the inverse Fourier transform to $\hat{u}(\omega, t)$. Using the formula for the inverse Fourier transform, we thus obtain that

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\omega, t) e^{i\omega x} \, d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \sqrt{\frac{\pi}{2}} e^{(1+i\omega)t} e^{i\omega x} \, d\omega \\ &= \frac{e^t}{2} \int_{-1}^{1} e^{i\omega(x+t)} \, d\omega \\ &= \frac{e^t}{2i(x+t)} e^{i\omega(x+t)} \Big|_{-1}^{1} \\ &= \frac{e^t}{2i(x+t)} \left(e^{i\omega(x+t)} - e^{-i\omega(x+t)} \right) \\ &= e^t \frac{\sin(x+t)}{x+t}. \end{aligned}$$

Problem 6. (4N only)

Consider the three-point finite-difference formula below:

$$\frac{-3u(x) + 4u(x+h) - u(x+2h)}{2h} = u'(x) + O(h^2) .$$

Using a computer with machine accuracy $\varepsilon > 0$, the approximation of u' will have, in addition to the **truncation error** $O(h^2)$, a **rounding error** dependent on ε and h. These two errors will have comparable orders of magnitude when

$$h = O(\varepsilon^p) , \quad p > 0.$$

Determine the value of p.

Solution.

Due to round-off errors, the function evaluations and operations in the *numerator* of the finite-difference formula will produce an error $O(\varepsilon)$. Since the finite difference has a $2h$ in the *denominator*, the actual error caused by round-off will then be $O(\varepsilon/h)$. Therefore, that error will be comparable to the truncation error $O(h^2)$ when

$$O(h^2) = O(\varepsilon/h)\,,$$

that is, when $h^3 = O(\varepsilon)$. We then have $h = O(\sqrt[3]{\varepsilon})$, so that $p = 1/3$.

Problem 6. (4D only)

Classify the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u\,\frac{\partial u}{\partial x}$$

as linear or nonlinear.

Show that, for given constants a and b,

$$u(x,t) = a + \frac{2}{at + x + b}$$

is a solution of the PDE.

Solution.

The PDE is nonlinear, due the $u\frac{\partial u}{\partial x}$ term.

To check the solution, use:

$$u_t = -\frac{2a}{(at+x+b)^2}, \qquad u_x = -\frac{2}{(at+x+b)^2}, \qquad u_{xx} = \frac{4}{(at+x+b)^3}.$$

Inserting this into the right hand side of the PDE gives

$$u_{xx} + uu_x = \frac{4}{(at+x+b)^3} + \left(a + \frac{2}{at+x+b}\right)\left(-\frac{2}{(at+x+b)^2}\right) = -\frac{2a}{(at+x+b)^2} = u_t$$

so $u(x,t)$ is a solution of the PDE.

Problem 7.

Consider the fixed-point equation $x = \sqrt{x} + 2$, which has a unique real solution r.

a) Show that the fixed-point iterations $x_{k+1} = \sqrt{x_k} + 2$ will converge to r, for any initial guess $x_0 \in [1, 9]$.

b) Starting from $x_0 = 2$, determine how many iterations, at most, will be needed until having

$$|x_{k+1} - r| \leq 2^{-10}\,.$$

Important: you are *not* being asked to perform these iterations!

Solution.

a) We have the fixed-point equation $x = g(x)$, where $g(x) = \sqrt{x} + 2$. The fixed-point theorem, which guarantees the existence of the unique root r and also the convergence of the fixed-point iterations, depend on properties of the function $g(x)$. We need to verify the following conditions:

   1) There exists a positive constant $L < 1$ so that $|g(x)'| \leq L$ for all $x \in [1, 9]$
   2) The function $g(x)$ stays within the interval of interest, that is: $g(x) \in [1, 9]$ for all $x \in [1, 9]$.

   To check the first one, we must differentiate $g(x)$:

$$g(x) = x^{\frac{1}{2}} + 2 \quad \Rightarrow g'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2\sqrt{x}}\,.$$

   Since $\sqrt{x} > 0$ for all $x > 0$, we have simply

$$|g'(x)| = g'(x) = \frac{1}{2\sqrt{x}}\,,$$

   which is of course a **strictly decreasing** function, because the square root function $\sqrt{x}$ in the denominator is increasing. Since $|g'(x)|$ is strictly decreasing, we know that its maximum value in the interval $x \in [1, 9]$ is simply $|g'(1)|$. Hence:

$$|g'(x)| \leq |g'(1)| = \frac{1}{2\sqrt{1}} = \frac{1}{2} \text{ for all } x \in [1, 9]\,.$$

   The first condition of the theorem is therefore met, with $L = 1/2$.

   Then, since $g(x)$ is clearly an **increasing function**, we know that its minimum and maximum values within the interval happen for $x = 1$ and $x = 9$, respectively:

$$g(1) \leq g(x) \leq g(9) \quad \Rightarrow 3 \leq g(x) \leq 5\,.$$

   Having $g(x) \in [3, 5]$ implies, in particular, $g(x) \in [1, 9]$, since the interval $[1, 9]$ contains $[3, 5]$. The last condition is thus fulfilled, which shows that the fixed-point iterations will convergence to the root r, for any initial guess $x_0 \in [1, 9]$.

b) As a consequence of the fixed-point theorem, we have the a-priori error estimate

$$|x_{k+1} - r| \leq \frac{L^{k+1}}{1-L}|g(x_0) - x_0|.$$

Therefore, if we find k such that

$$\frac{L^{k+1}}{1-L}|g(x_0) - x_0| = 2^{-10},$$

then we know for sure that we will also have $|x_{k+1} - r| \leq 2^{-10}$. We therefore have to solve the equation above for k, with $L = 1/2$, $x_0 = 2$ and $g(x_0) = \sqrt{x_0} + 2 = \sqrt{2} + 2$. This gives us

$$2^{-10} = \frac{(1/2)^{k+1}}{1-1/2}|(\sqrt{2}+2) - 2| = \left(\frac{1}{2}\right)^k |\sqrt{2}| \quad \Leftrightarrow \quad 2^{-k} = 2^{-10.5} \quad \Leftrightarrow \quad k = 10.5\,.$$

Since we cannot perform "half an iteration", we must round k up to 11.

Problem 8.

A Runge–Kutta method for solving a scalar ODE, $y' = f(t, y)$, is implemented, the main block of the code is given below:

```
for n in range(N):
    yn = y[n]
    tn = t[n]
    k1 = f(tn, yn)
    k2 = f(tn+0.5*h,yn+0.5*h*k1)
    k3 = f(tn+h, yn+0.5*h*(k1+k2))
    y[n+1] = yn+h/6*(k1+4*k2+k3)
```

Write down the Butcher-tableau, and determine the order of the method.

Solution.

The Butcher-tableau is given by

$$\begin{array}{c|ccc} 0 & & & \\ 1/2 & 1/2 & & \\ 1 & 1/2 & 1/2 & \\ \hline & 1/6 & 2/3 & 1/6 \end{array}$$

Check the order conditions, using that $c_1 = 0$.

$p = 1$	$b_1 + b_2 + b_3 = \frac{1}{6} + \frac{2}{3} + \frac{1}{6} = 1$	OK
$p = 2$	$b_2 c_2 + b_3 c_3 = \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{6} \cdot 1 = 1/2$	OK
$p = 3$	$b_2 c_2^2 + b_3 c_3^2 = \frac{2}{3} \cdot \frac{1}{4} + \frac{1}{6} \cdot 1 = 1/3$	OK
$p = 3$	$b_3 a_{32} c_2 = \frac{1}{6} \cdot \frac{1}{2} \cdot \frac{1}{2} \neq 1/6$	

The last order 3 condition is not satisfied, so the method is of order 2 only.

Problem 9.

We are given the following embedded Runge-Kutta pair:

$$\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 \\
3/4 & 0 & 3/4 & 0 \\
\hline
 & 0 & 1 & 0 \\
 & 2/9 & 1/3 & 4/9
\end{array}.$$

The first row of b coefficients gives a second order accurate method, and the second row gives a method of order three. The pair is applied to the ODE

$$y' = -y^2, \quad y(0) = 1.$$

a) Let the initial step size be $h_0 = 0.2$ and perform one step with the highest order method.

b) Let $y_1, \hat{y}_1$ denote the solutions after one step with stepsize $h_0 = 0.2$ with the lowest, respectively the highest order method, both starting from $y(0) = y_0 = 1$.

   Compute the local error estimate $\hat{\epsilon}_1 = |y_1 - \hat{y}_1|$.

c) Comparing $\hat{\epsilon}_1$ with the tolerance $\text{tol} = 10^{-3}$, check if the first step is acceptable, and if not, compute a new stepsize h_{new}. Use the pessimist factor (safety factor) $P = 0.8$.

Solution.

The method applied to a general ODE $y' = f(t, y)$ is given by

$$
\begin{aligned}
k_1 &= f(t_n, y_n)\\
k_2 &= f(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1)\\
k_3 &= f(t_n + \frac{3}{4}h, y_n + \frac{3}{4}hk_2)\\
y_{n+1} &= y_n + hk_2\\
\hat{y}_{n+1} &= y_n + \frac{1}{9}h(2k_1 + 3k_2 + 4k_3)
\end{aligned}
$$

and the error estimate is given by

$$\hat{\epsilon}_{n+1} = h|(b_1 - \hat{b}_1)k_1 + (b_2 - \hat{b}_2)k_2 + (b_3 - \hat{b}_3)k_3| = h|\frac{2}{9}k_1 - \frac{2}{3}k_2 + \frac{4}{9}k_3|.$$

a) Applied to $y' = -y^2$, $y(0) = y_0 = 1$ this becomes

$$k_1 = -1.0, \quad k_2 = -0.81, \quad k_3 = -0.77176225$$

and

$$\hat{y}_1 = 0.83295447.$$

b) The local error estimate is

$$\hat{\epsilon}_1 = 5.045 \cdot 10^{-3}.$$

c) Since $\hat{\epsilon}_1 = 5.045 \cdot 10^{-3} > \text{Tol} = 10^{-3}$, the step will be rejected and the step size reduced. Since the order of the lowest order method is 2, the step size is modified by

$$h_{\text{new}} = P\left(\frac{\text{Tol}}{\hat{\epsilon}_1}\right)^{\frac{1}{p+1}} h = P\sqrt[3]{\frac{\text{Tol}}{\hat{\epsilon}_1}}\, h.$$

With $P = 0.8$, this becomes

$$h_{\text{new}} = 0.09324.$$

Problem 10.

Given the two point boundary value problem

$$u'' + x^2 u = \sin(\pi x), \qquad 0 \le x \le 1,$$

with boundary conditions

$$u'(0) = 0 \quad \text{and} \quad u(1) = 0.$$

a) Set up a finite difference scheme for this problem using a constant grid size $h = 1/N$.

b) An attempt of an implementation is given below:

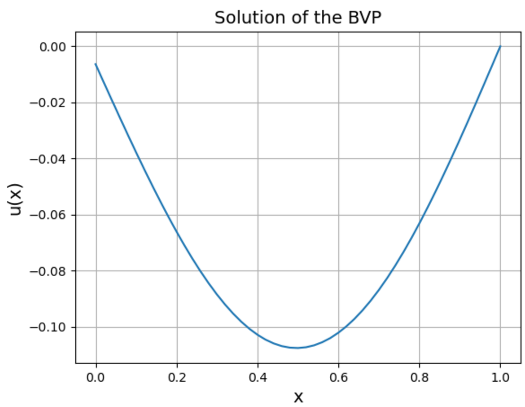
```
import numpy as np

N = 50
h = 1/N
A = np.zeros((N,N))
b = np.zeros(N)
U = np.zeros(N+1)
x = np.linspace(0,1,N+1)

# The coefficient matrix A and the right hand side b
A[0,0] = -2
A[0,1] = 1
for i in range(1,N-1):
    xi = x[i]
    A[i,i-1] = 1
    A[i,i] = -2+h**2*xi**2
    A[i,i+1] = 1
    b[i] = np.sin(np.pi*xi)*h**2
A[N-1,N-2] = 1
A[N-1,N-1] = -2+h**2

U[:-1] = np.linalg.solve(A,b)  # U[N] = 0 is known
```

The plot of the numerical solution is:



How can you conclude that this is *not* the correct solution?

Locate the error in the code.

Solution.

a) Let $h = 1/N$, $x_i = ih$, $i = 0, 1, \ldots, N$. Using a central difference scheme for the second derivative gives

$$\frac{u(x_i + h) - 2u(x_i) + u(x_i - h)}{h^2} + O(h^2) + x_i u(x_i) = \sin(\pi x_i), \qquad i = 1, \cdots, N-1$$

and $u(1) = 0$. For the left side boundary, use the trick with a false boundary, that is, combine

$$u'(0) = \frac{u(h) - u(-h)}{2h} + O(h^2) = 0,$$

with the equation above for $i = 0$. Replace $u(x_i)$ with its approximations U_i, multiply by h^2, and the finite difference scheme for this problem becomes:

$$\begin{aligned} -2U_0 + 2U_1 &= 0, \\ U_{i-1} - (2 - h^2 x_i^2)U_i + U_{i+1} &= \sin(\pi x_i), \qquad i = 1, \ldots, N-1, \\ U_N &= 0. \end{aligned}$$

b) Clearly, the left side boundary condition $u'(0) = 0$ is not satisfied. It is then reasonable to assume that the error has to be in the implementation of this boundary condition, which is in the first of the equations in the scheme (for $i = 0$), corresponding to the first row of the coefficient matrix A. The error is in line 12, it should be `A[0,1]=2`.

List of Fourier Transforms.

$f(x)$	$\hat{f}(\omega)$
e^{-ax^2}	$\frac{1}{\sqrt{2a}} e^{-\frac{\omega^2}{4a}}$
$e^{-a\|x\|}$	$\sqrt{\frac{2}{\pi}} \frac{a}{\omega^2 + a^2}$
$\frac{1}{x^2 + a^2}$ for $a > 0$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\|\omega\|}}{a}$
$\begin{cases} 1 & \text{for } \|x\| < a \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(\omega a)}{\omega}$
$\sqrt{\frac{2}{\pi}} \frac{\sin(ax)}{x}$	$\begin{cases} 1 & \text{for } \|\omega\| < a \\ 0 & \text{otherwise} \end{cases}$

List of Laplace Transforms.

$f(t)$	$F(s)$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2 - \omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2 - \omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s - a}$
$f(t-a)u(t-a)$	$e^{-sa}F(s)$
$\delta(t-a)$	e^{-sa}

Trigonometric identities.

- $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
- $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
- $\sin\alpha\cos\beta = \frac{1}{2}\big(\sin(\alpha - \beta) + \sin(\alpha + \beta)\big)$
- $\cos(2\alpha) = 2\cos^2(\alpha) - 1 = 1 - 2\sin^2(\alpha)$
- $2\sin\alpha\cos\beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)$
- $2\cos\alpha\sin\beta = \sin(\alpha + \beta) - \sin(\alpha - \beta)$
- $2\cos\alpha\cos\beta = \cos(\alpha - \beta) + \cos(\alpha + \beta)$
- $2\sin\alpha\sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$

Integrals.

- $\int x^n \cos ax \, dx = \frac{1}{a}x^n \sin ax - \frac{n}{a}\int x^{n-1} \sin ax \, dx$
- $\int x^n \sin ax \, dx = -\frac{1}{a}x^n \cos ax + \frac{n}{a}\int x^{n-1} \cos ax \, dx$

Order conditions for Runge-Kutta methods

p	Conditions	p	Conditions
1	$\sum_{i=1}^{s} b_i = 1$	4	$\sum_{i=1}^{s} b_i c_i^3 = \frac{1}{4}$
2	$\sum_{i=1}^{s} b_i c_i = \frac{1}{2}$		$\sum_{i=1}^{s}\sum_{j=1}^{s} b_i c_i a_{ij} c_j = \frac{1}{8}$
3	$\sum_{i=1}^{s} b_i c_i^2 = \frac{1}{3}$		$\sum_{i=1}^{s}\sum_{j=1}^{s} b_i a_{ij} c_j^2 = \frac{1}{12}$
	$\sum_{i=1}^{s}\sum_{j=1}^{s} b_i a_{ij} c_j = \frac{1}{6}$		$\sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{k=1}^{s} b_i a_{ij} a_{jk} c_k = \frac{1}{24}$