Exercise #9

13. March 2023

Problem 1.

In this problem, we consider the numerical solution of the heat equation

$$u_t = u_{xx} \qquad \text{for } 0 < x < 1 \text{ and } t > 0$$

with the initial condition

$$u(x,0) = f(x) \qquad \text{for } 0 \le x \le 1$$

for some given $f\colon [0,1] \longrightarrow \mathbb{R}$, and the mixed boundary conditions

$$\partial_x u(0,t) = a(u(0,t) - g(t)) \quad \text{and} \quad \partial_x u(1,t) = p(t) \qquad \text{for } t > 0,$$

where $a > 0$ is some fixed parameter, $g\colon \mathbb{R}_{>0} \longrightarrow \mathbb{R}$ and $p\colon \mathbb{R}_{>0} \longrightarrow \mathbb{R}$ are given functions.

This models for instance the temperature in a thin, insulated rod, with heat transfer through both endpoints. At $x = 0$, we can understand $g(t)$ as describing the outside temperature and a as the heat transfer coefficient. At $x = 1$, $p(t)$ could model a controlled heating or cooling of the rod endpoint.

a) Set up an *explicit* finite difference scheme for the solution of this PDE. The final scheme should be written in the form

$$\begin{aligned}
U_i^0 &= \dots && \text{for } i = 0, \dots, M, \\
U_0^{n+1} &= \dots && \text{for } n = 0, 1, \dots, \\
U_i^{n+1} &= \dots, && \text{for } i = 1, \dots, M-1, \text{ and } n = 0, 1, \dots, \\
U_M^{n+1} &= \dots, && \text{for } n = 0, 1, \dots,
\end{aligned}$$

where the ... consist of given and/or previously computed values.

b) Set up an *implicit* finite difference scheme for the solution of this PDE. The final scheme should be written in matrix-vector form as

$$\begin{aligned} U_i^0 &= \dots && \text{for } i = 0, \dots, M, \\ C\vec{U}^{n+1} &= D\vec{U}^n + \dots && \text{for } n = 0, 1, \dots, \end{aligned}$$

where $C, D \in \mathbb{R}^{(M+1)\times(M+1)}$ are matrices and ... consist of given values.

c) Consider in particular the case where $a = \frac{\pi}{2}$, $g(t) = \frac{3}{2}e^{-\frac{\pi^2}{4}t}$ and $p(t) = -\frac{\pi}{2}e^{-\frac{\pi^2}{4}t}$ for all $t > 0$, and

$$f(x) = \left(\cos\left(\frac{\pi}{2}x\right) - \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\right).$$

Verify that the function

$$u(x,t) = \left(\cos\left(\frac{\pi}{2}x\right) - \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\right)e^{-\frac{\pi^2}{4}t}$$

is a solution of the PDE with these boundary and initial conditions.

d) (J) Implement the schemes from parts a) and b), and test them for the initial and boundary conditions of part c). Plot your numerical approximations for $t = 0$, 0.1, 1.0, and 5.0, and compare them with the exact solution.

Consider specifically your explicit scheme with step size $h = 0.1$. For which time step length k (approximately) does the solution become unstable?

You may use the jupyter templates that are provided together with the exercise sheet.

Note: The main task in parts a) and b) of this exercise is to find a correct way of incorporating the mixed boundary conditions in the numerical methods. You answer should include a detailed explanation of how you treat these boundary conditions, both in the explicit and the implicit case. The other steps in the discretisation, which are really the same as in the lecture / lecture notes can be less detailed.

Problem 2.

Use the Fourier transform to find the function $u(x,t)$ satisfying the PDE

$$u_t = tu_{xx} \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0$$

with the initial condition

$$u(x,0) = e^{-2x^2} \qquad \text{for } x \in \mathbb{R}.$$

Problem 3.

Use the Fourier transform to find the function $u(x,t)$ satisfying the *wave equation* on an infinite string

$$u_{tt} = u_{xx} \qquad \text{for } -\infty < x < \infty \text{ and } t > 0$$

with the initial conditions

$$\left.\begin{aligned} u(x,0) &= \frac{1}{x^2+1}, \\ \partial_t u(x,0) &= 0 \end{aligned}\right\} \qquad \text{for } x \in \mathbb{R}.$$

The next exercises are optional and should not be handed in!

Problem 4.

Use the Fourier transform to find the function $u(x,t)$ satisfying the *telegraph equation*

$$u_{tt} + 2u_t + u = u_{xx} \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0$$

with the initial conditions

$$\left.\begin{aligned} u(x,0) &= \operatorname{sinc}(x), \\ \partial_t u(x,0) &= -\operatorname{sinc}(x) \end{aligned}\right\} \qquad \text{for } x \in \mathbb{R}.$$

Here the sinc function is defined as

$$\operatorname{sinc}(x) = \begin{cases} \dfrac{\sin(x)}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$