Exercise #8

6. March 2023

Problem 1. (Finite difference method for the wave equation)
Consider the wave equation

$$u_{tt} = 9u_{xx}, \qquad -2 \le x \le 2, \quad t \ge 0, \tag{1a}$$
$$u(-2,t) = u(2,t) = 0, \qquad t \ge 0, \tag{1b}$$
$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \qquad -2 \le x \le 2. \tag{1c}$$

a) Find the exact solutions of the wave equation for the following two sets of initial conditions:

- The conditions
$$f(x) = \sin\left(\frac{\pi x}{2}\right) \quad \text{and} \quad g(x) = 0. \tag{2}$$

- The conditions
$$f(x) = 0 \quad \text{and} \quad g(x) = \pi \sin\left(\frac{\pi x}{2}\right). \tag{3}$$

b) - Set up a finite difference scheme, using central differences for the second derivatives in both directions. Use step sizes h in space and k in the temporal direction. Notice that special care has to be taken for the first temporal step. The final scheme should be written in the form

$$U_i^1 = \cdots, \qquad i = 1, \ldots, M-1,$$
$$U_i^{n+1} = \cdots, \qquad i = 1, \ldots, M-1, \qquad n = 1, 2, \ldots,$$

where the $\cdots$ consist of given and/or previously computed values.

NB! Your solution should include an explanation on how the explicit scheme is obtained.

- Use the initial values (2) Let $h = 1$ and $k = \frac{1}{3}$ and use the formulas above to find approximations to $u(x_i, \frac{1}{3})$, $i = 1, \ldots, 3$ (by hand!).

c) Implement and test the scheme derived in point (b). A template for the implementation can be found in the Jupyternote `PDE_wave_fdm.ipynb`.

It is more interesting to study the animations of the simulations, however, for the hand-in, you are asked to plot the solutions for $t = 0, 0.4, 1.2$ and 2 if nothing else is said.

- Use the initial values from (2). Let $h = 0.1$ and $k = 0.01$. Plot the solutions, and compare with the exact solution.

  You may also like to try (but not hand in) with k just a tiny bit larger than 0.01, and observe what happens in this case. You should observe instabilites.

The next bullets points are optional and should not be handed in

- Repeat the experiment above with the initial values given by (3).

- Use the initial values (2), but let now the wave speed c be given by

$$c(x) = \begin{cases} 1 & \text{for } x \leq 0, \\ 2 & \text{for } x > 0. \end{cases}$$

  Plot the solution for $t = 0, 0.4, 1.2$ and 2.

Problem 2.

Use D'Alembert's method to find the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$$

for $t > 0$ and all $x \in \mathbb{R}$ with initial conditions

$$u(x,0) = \sin x \qquad \text{and} \qquad \frac{\partial u(x,0)}{\partial t} = e^{-x/4}.$$

Problem 3. (1D Heat Equation)

Solve the differential equation

$$\begin{cases} \dfrac{\partial}{\partial t}u - \frac{1}{4}\dfrac{\partial^2}{\partial x^2}u = 0, \\ u(0,t) = u(\frac{\pi}{2},t) = 0, \\ u(x,0) = 2x(x - \frac{\pi}{2}) \end{cases}$$

for $x \in [0, \frac{\pi}{2}]$ and $t \in [0, \infty]$.

The next exercises are optional and should not be handed in!

Problem 4.

Use D'Alembert's method to find the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

for $t > 0$ and all $x \in \mathbb{R}$ with initial conditions

$$u(x,0) = \cos x \qquad \text{and} \qquad \frac{\partial u(x,0)}{\partial t} = xe^{-x^2}.$$

Problem 5.

Vibrations in a stiff metal rod can be roughly described by the fourth order PDE

$$\frac{\partial^2 u}{\partial t^2} = -c^2 \frac{\partial^4 u}{\partial x^4}.$$

We assume now that the metal rod is simply supported at both ends at $x = 0$ and $x = \pi$, and has zero curvature at both ends. This corresponds to the boundary conditions

$$u(0,t) = u(\pi,t) = 0 \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(\pi,t) = 0$$

for all $t > 0$.

a) We use the idea of separation of variables and consider solutions of the form $u(x,t) = F(x)G(t)$. Derive ordinary differential equations for the functions F and G.

b) Verify that all functions of the form

$$F(x) = A\sin(\beta x) + B\cos(\beta x) + C\sinh(\beta x) + D\cosh(\beta x)$$

with $\beta > 0$ satisfy the ODE for F derived in the previous point. For which values of $\beta > 0$ are the boundary conditions satisfied? What are the corresponding solutions of the equation for G?