Exercise #5

13. February 2023

Problem 1. (Trigonometric series)

Determine the coefficients $a_0, a_n, b_n,\ n \in \mathbb{N}$ of the trigonometric Fourier series $f \sim a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi}{L}x) + b_n \sin(\frac{n\pi}{L}x)$ for the follwing functions

a) $f(x) = 1/2 + \cos(2x) - 4\sin(4x)$, (find the fundamental period yourself);

b) $f(x) = 1 + \sin^2(x)$, which has period $T = 2L = \pi$;

c) $f(x) = 1 - x$ for $-1 \leq x \leq 1$, with period $T = 2L = 2$;

d) $f(x) = |\cos(3\pi x)|$, that has period $T = 2L = 1/3$.

Problem 2. (Complex series)

Determine the coefficients c_k of the complex Fourier series $f \sim \sum_{k=-\infty}^{\infty} c_k e^{i\frac{k\pi}{L}x}$ for the following functions

a) $f(x) = -3e^{-4ix} + e^{-ix} + 4e^{3ix}$, (indentify the fundamental period yourself),

b) $f(x) = \cos^2(x)$, for $-\pi/2 \leq x \leq \pi/2$, with period $T = 2L = \pi$,

c) $f(x) = x^2 - x$ for $-1 \leq x \leq 1$, with period $T = 2L = 2$,

d) $f(x) = \mathrm{e}^{-x}$ for $-1 \leq x \leq 1$, with period $T = 2L = 2$,

Problem 3. (Parseval's theorem)

Consider two periodic functions $f(x)$ and $g(x)$ and their truncated Fourier series $f_N(x)$ and $g_N(x)$. The errors E_N between each function and its trigonometric approximation can be computed using Parseval's identity.

a) What does the formula for E_N looks like?

b) Let $f(x) = \mathrm{e}^{-x}$ for $-1 \leq x \leq 1$, with period $T = 2L = 2$. Based on the Fourier coefficients of $f(x)$, calculate the error E_N for $N = 2$, 4 and 8.
Hint: you have already computed the (complex) coefficients for this function on for Problem 2 (d).

c) Now, do the same for $g(x) = \mathrm{e}^{-|x|}$, $-1 \leq x \leq 1$, with period $T = 2L = 2$.

d) Why is the error so much larger when approximating $f(x)$, than in the case of $g(x)$?

Problem 4. (Gibbs phenomenon)

In this exercise, we will illustrate what happens when we try to use Fourier series to describe a discontinuous function $f(x)$. Let's consider $1 - x^2$, $x \in [0, 1]$, and its odd extension to the interval $[-1, 0]$, $f(x) = sign(x)(1 - x^2)$ which has a discontinuity at $x = 0$ and $L = 1$. As usual, the truncated trigonometric series can be written as

$$S_N(x) = a_0 + \sum_{n=1}^{N} a_n \cos n\pi x + b_n \sin n\pi x\,, \quad n \in \mathbb{N}\,.$$

We will numerically investigate what happens to $S_N(x)$ when N is chosen larger and larger.

a) Determine an expression for the trigonometric Fourier coefficients a_0, a_n and b_n.

b) Using the Jupyter notebook `Gibbs.ipynb`, plot $f(x)$ and $S_N(x)$ for $N = 5, 20$ and 100.

c) We can easily verify that, at $x = 0$, the function's value jumps from -1 to 1. Let's define the global error as

$$\varepsilon_N := \max\{|S_N(x) - f(x)|\}\,, \quad x \in [-1, 1]\,,$$

as already implemented in the same Jupyter notebook. Following that, compute ε_N for $N = 1000$. In percentual terms, how large is this error with respect to the jump height 2?

The next exercises are optional and should not be handed in!

Problem 5. (Convolution)

a) Compute the complex Fourier series of the 2π-periodic function

$$f(x) = \sin(3x) + 5\cos(2x).$$

b) Compute the complex Fourier series of the 2π-periodic function

$$g(x) = x^2, \quad -\pi \leq x \leq \pi,$$

c) Show that the convolution $h(x) = f(x) * g(x)$ is given as

$$h(x) = -\frac{4\pi}{9}\sin(3x) + 5\pi\cos(2x).$$

d) Compute the complex Fourier series of $h(x)$.

Hint. You can reuse the results from a) and b) and you do not have to solve integrals.

Problem 6. (Infinite sums)

In this exercise, we will use Fourier series and coefficients to show a few identities.

a) Consider the 2π-periodic function $f(x) = \begin{cases} -\pi - x & \text{if } -\pi < x < -\frac{\pi}{2}, \\ x & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2}, \\ \pi - x & \text{if } \frac{\pi}{2} < x \leq \pi. \end{cases}$

Compute its Fourier coefficients and use the result, together with Parseval's identity, to show that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.$$

b) Consider the function $f(x) = x$, defined for $x \in [0, 1]$. Compute the Fourier series of its *even extension*, and use the result to show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \ldots = \frac{\pi^2}{8}.$$

c) Now, considering the same function $f(x)$ from (b), compute the Fourier series of its *odd extension* and use the result to show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots = \frac{\pi}{4}.$$

Problem 7. (Derivative)
Let $c_k(f)$ denote the complex Fourier coefficients for the twice continuously differentiable and 2π-periodic function $f(x)$. Show that the complex Fourier coefficients $c_k(f')$ for $f'(x)$ are given by $c_k(f') = ikc_k$.