Exercise #4

6. February 2023

Problem 1. (Bisection Method)
Consider the function

$$f(x) = (1-x)2^{x+2} + (2^x - 2)x^2 + 8(x-1),$$

we want to find roots of $f(x)$ on the interval $[-3, 3]$.

a) Plot the function on the interval $[-3, 3]$.

b) Do 4 iterations of the bisection method by hand. What root(s) did you find and what is the error bound?

c) How many iterations are required to guarantee an error smaller than 10^{-3}?

d) Use python to find roots of $f(x)$ with an error smaller than 10^{-3}.

Problem 2. (Radius of Convergence)
Consider the function

$$f(x) = x\log(x) - x.$$

a) Show that the minimal function $M(a, b)$ such that

$$\left|\frac{f''(y)}{f'(x)}\right| \leq 2M(a, b),$$

for all $x, y \in [a, b]$, where $0 < a < b < \infty$, is given by

$$M(a, b) = \frac{1}{2a \log(a)}, \quad \text{if } a > 1,$$

and that no such function exists if $a \leq 1$.

b) Use the convergence theorem to show that Newton's method converges for x_0 satisfying both of the two inequalities

$$\begin{aligned} 2(e - |x_0 - e|) \log(e - |x_0 - e|) \geq& |x_0 - e|, \\ |x_0 - e| <& |e - 1|, \end{aligned}$$

where e is Euler's number.

c) Conclude how to find for which x_0 Newton's method is guaranteed to converge for $f(x)$, by looking at the fix points of the function

$$g(x) = 2(e - x) \log(e - x).$$

Can fixed point iterations of this function g be used to find this fixed point?

d) Use the bisection method to find the fixed point of $g(x)$. For which x_0 is Newton's method guaranteed to converge for $f(x)$?

e) Verify numerically that Newton's method converges for some x_0 that you found in part d), also test some x_0 that were not found in part d).

Problem 3. (Periodic functions)

Recall that a function $f\colon \mathbb{R} \to \mathbb{R}$ is called periodic, if there exists $p > 0$ (a period of f) such that

$$f(x + p) = f(x) \text{ for all } x \in \mathbb{R}.$$

Moreover, the smallest positive number for which this statement holds (if it exists), is called the fundamental period of f.

a) Decide whether the following statement is true or false, and then find either a proof or a counterexample: Every periodic function has a fundamental period.

b) What is the fundamental period of the following functions:

- $f(x) = \sin(x+2)$
- $f(x) = \pi\cos(\pi x)$
- $f(x) = \cos(\frac{2\pi}{m+1}x) + \sin(\frac{2\pi}{n-1}x), \quad n, m \in \mathbb{N} \setminus \{1\}$.

Problem 4.

Recall that a function f is even if $f(-x) = f(x)$, and odd if $-f(-x) = f(x)$.

a) Show that if $f(x)$ is odd, and $g(x)$ is even, then $h(x) := f(x)g(x)$ is odd.

b) Show that if $f(x)$ is odd, then $h(x) := f^2(x)$ is even.

c) Show that if $f(x)$ is odd, then for any $L > 0$,

$$\int_{-L}^{L} f(x)dx = 0.$$

d) Show that if $f(x)$ is odd and $g(x)$ is even then $f \circ g$ and $g \circ f$ are both even.

e) Show that if $f(x)$ is even, then for any $L > 0$,

$$\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx.$$

Problem 5. (Trigonometric interpolation)

In our first lectures, we saw how polynomials can be used to interpolate functions or discrete datasets. That is actually not the only possibility, and in fact we can also use trigonometric functions for interpolation. Namely, the set $\{1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots\}$ is a common basis for that.

Let $y = f(x) = 2 + \sin(x)\cos(x)$ be the function we wish to interpolate, for which we consider three data points:

i	0	1	2
x_i	0	$\pi/6$	$\pi/3$
y_i	2	$2+\sqrt{3}/4$	$2+\sqrt{3}/4$

Our task here is to construct a trigonometric function $p(x)$ that goes through these three points. Since $f(x)$ is just a constant plus an odd function, it suffices to use only sinus terms plus the constant for the interpolation. That is, we are looking for some

$$p(x) = \alpha_1 + \alpha_2 \sin x + \alpha_3 \sin 2x$$

such that $p(x_i) = y_i$ for $i = 0, 1, 2$, with α_0, α_1 and α_2 being three real coefficients to be determined.

a) Based on the data in table, compute the interpolation coefficients α_0, α_1 and α_2.

b) Show that, for this particular case, we have $p(x) = f(x)$ for all $x \in \mathbb{R}$, that is, the interpolant is identical to the function being interpolated.
Hint: remember the trigonometric identity $\sin 2x = 2 \sin x \cos x$.