Exercise #1

16. January 2023

Exercises marked with a (J) should be handed in as a Jupyter notebook.

Optional exercises will not be corrected.

Problem 1. (Taylor Polynomials)

a) Compute all Taylor polynomials of $f(x) = x^4 + 2x^3 + x^2 + 5$ around $x_0 = 1$

b) Compute the Taylor series of $g(x) = \ln(1+x)$ around $x_0 = 0$

Problem 2. (Taylor expansion error)

Show that for a function $f \in C^l([0,1];\mathbb{R})$, we have the following expression for the remainder of the Taylor expansion (about 0, evaluated at 1),

$$f(1) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} = \int_0^1 \frac{m}{m!} s^{m-1} f^{(m)}(1-s)\,\mathrm{d}s,$$

for every $m \leq l$.

Hint. Use induction. Check that it holds for $m = 1$, and then try to manipulate the expression,

$$f(1) - \sum_{k=0}^{m} \frac{f^{(k)}(0)}{k!} = f(1) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} - \frac{f^{(m)}(0)}{m!},$$

so that it is equal to $\int_0^1 \frac{m+1}{(m+1)!} s^m f^{(m+1)}(1-s) ds$.

Problem 3. (Numerical differentiation)

We want to find a difference formula of the form

$$u''(x) \approx \frac{1}{h^2}\left(a_1 u(x) + a_2 u(x-h) + a_3 u(x - \frac{1}{2}h)\right),$$

where a_1, a_2 and a_3 are constants to be determined.

Find the coefficients a_1, a_2 and a_3 which makes this scheme convergent. Find an expression for the error term.

Problem 4. (A boundary value problem)

Given the two point boundary value problem:

$$u_{xx} + 2u_x + \pi^2 u = \cos(\pi x) - \pi(x+1)\sin(\pi x), \quad 0 \le x \le 2, \qquad u(0) = 0, \qquad u(2) = 1$$

a) Verify that the exact solution is

$$u(x) = \frac{x}{2}\cos(\pi x).$$

b) Set up a finite difference scheme for this problem, using central differences. Use $\Delta x = 2/N$ as the grid size, and let $x_i = i\Delta x$, $i = 0, 1, \dots, N$.

c) Let $N = 4$ and use the above formula to find approximations $U_i \approx u(x_i)$, $i = 1, 2, 3$. (That is: Set up the system of equations, and solve it). Compare with the exact solution.

d) (J) Modify the code `Example 1, BVP` in the note on boundary value problems, and solve the problem numerically. Use $N = 10, 20, 40$ in your simulation. For each N, write down the error

$$e(h) = \max_{i=0,\dots,N} |u(x_i) - U_i|.$$

What can you deduce about the order of the scheme from this experiment?

Optional Exercises.
The next two exercises are optional and should not be handed in

Problem 5. (A boundary value problem with variable coefficients)

Given the two point boundary value problem:

$$u_{xx} - \frac{2}{x}u_x + \frac{2}{x^2}u = -\frac{x\pi^2}{2}\cos(\pi x), \quad 1 \le x \le 2, \qquad u(1) = -\frac{1}{2} \qquad u(2) = 1$$

a) Verify that the exact solution is

$$u(x) = \frac{x}{2}\cos(\pi x).$$

b) Set up a finite difference scheme for this problem, using central differences. Use $\Delta x = 1/N$ as the grid size, and let $x_i = i\Delta x$, $i = 0, 1, \dots, N$.

c) Let $N = 4$ and use the above formula to find approximations $U_i \approx u(x_i)$, $i = 1, 2, 3$. (That is: Set up the system of equations, and solve it). Compare with the exact solution.

d) (J) Modify the code `Example 1, BVP` in the note on boundary value problems, and solve the problem numerically. Use $N = 10, 20, 40$ in your simulation. For each N, write down the error

$$e(h) = \max_{i=0,\dots,N} |u(x_i) - U_i|.$$

What can you deduce about the order of the scheme from this experiment?

Problem 6. (a *nonlinear* boundary value problem)

Given the two point boundary value problem:

$$\frac{1}{10}u'' + u' + u^2 = 0, \quad u(0) = 1, \quad u(1) = 0.$$

a) Set up a finite difference scheme for this problem, using central differences. Use $\Delta x = 1/N$ as the grid size, and let $x_i = i\Delta x$, $i = 0, 1, \dots, N$. You end up with a system of nonlinear equations.

b) (J). Solve the problem in python. Use $N = 50$ (although you may try other values as well).

Hint: Use the python function `scipy.optimize.fsolve`. Read the documentation to figure out how to use it.