NTNU

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TMA4125 Matematikk 4N

Revision / Summary

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1. Introduction and Preliminaries

Let's do a short recapitulation of the topics we covered in TMA4125 In the first week we discussed a few concepts from earlier classes

- What Mathematics 4N is about
- real and complex vector spaces, e. g. of polynomials
- norm, scalar product and orthogonality
- Convergence, its order and their numerical verification
- Taylor expansion, $\mathcal{O}$ notation
- An introductory boundary value problem (BVP)

These concepts will not directly be asked in the exam, but we use them often.

The following slides state the key concepts/topics we covered.

2. Interpolation

Goal. Find a function f that describes this data, i. e.

$$f(x_i) = y_i \qquad i = 0, \dots, n,$$

- Statement of the Interpolation problem
- Lagrange interpolation based on Lagrange Polynomials (Cardinal functions)
- Estimate the interpolation Error.
- Runge's example and optimal distribution of interpolation nodes (Chebyshev nodes/interpolation)
- B-Spines, especially
  - the space dimension of $S_{k,\Delta}$
  - cubic splices ($k = 3$)
  - boundary conditions.

2. Numerical Integration

Goal. Solve the task of integration by using a numerical quadrature

$$I[f](a,b) := \int_a^b f(x)\,\mathrm{d}x \approx Q[f](a,b) = \sum_{i=0}^{n} w_i f(x_i),$$

using certain points $x_i \in [a,b]$ and weights w_i.

- Midpoint rule, trapezoidal rule, Simpson rule, (Gauß-Legendre)
- Quadrature from integrating interpolation polynomials
- Degree of precision, error estimates
- Composite Quadrature
- Adaptive Integration
- Error Estimates

3. Numerical errors & Fix point iteration

In general we discussed

- Number representations and its limitations (over-, underflow, rounding)
- machine precision
- rounding errors for finite differences

Goal. Numerically find the solution to a nonlinear equation

$$f(x) = 0 \quad \text{or for the multivariate case} \quad \boldsymbol{f}(\boldsymbol{x}) = \mathbf{0}.$$

- Existence and uniqueness of solutions
- Bisection Method
- Fix point iteration: Existence, Uniqueness, Convergence

4. Nonlinear Equations: Newton's Method

Goal. Numerically find the solution to a nonlinear equation

$$f(x) = 0 \qquad \text{or for the multivariate case } \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}.$$

As a second equation we discussed Newton's method.

- in the one-dimensional case
- in the multivariate case involving the Jacobian
- Convergence
- Error Analysis

5. Fourier Series

Goal. Describe a (2π-)periodic function f as

$$f \sim \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathrm{i}kx} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

- composing and decomposing functions
- trigonometric polynomials, vector spaces
- Orthogonal projection and best approximation (again)
- Convergence of the Fourier series, Gibbs phenomenon
- Real and complex Fourier Series, Sine and Cosine series
- $2L$-periodic functions, even & odd extension
- Parseval identity, Spektrum and Amplitude
- Convolution, Derivatives & further properties

6a. (Continuous) Fourier Transform

Goal. Extend Fourier Series idea to the whole real line

$$\hat{f}(\omega) = \mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-\mathrm{i}\omega x} \,\mathrm{d}x$$

similarly: Fourier Sine and Fourier Cosine transform

- derivation of the (Continuous) Fourier Transform
- Linearity, Derivatives, Convolution
- Inverse Fourier Transform

6b. Discrete Fourier Transform

Goal. Compute Fourier Series on the computer: Discrete Fourier Transform (DFT), i. e. for a signal $\boldsymbol{f} = (f_0, \ldots, f_{N-1})^{\mathrm{T}} \in \mathbb{C}^N$ compute

$$\hat{f}_k = \sum_{j=0}^{N-1} f_j \, \mathrm{e}^{-2\pi \mathrm{i} jk/N}, \qquad k = 0, \ldots, N-1.$$

or shorter in matrix-vector form

$$\hat{\boldsymbol{f}} = \mathcal{F}_N \boldsymbol{f} \quad \text{with } \mathcal{F}_N = \left(\mathrm{e}^{-2\pi \mathrm{i} jk/N}\right)_{j,k=0}^{N-1} = \left(w_N^{jk}\right)_{j,k=0}^{N-1}$$

- Aliasing
- FFT-shift and interpretation of discrete Fourier coefficients
- Application: Denoising / Filtering out a frequency.
- Discrete Sine and Cosine Transforms, FFT (short)

7. Wave Equation

$$
\begin{cases}
\frac{\partial^2}{\partial t^2} u = c^2 \Delta u & \boldsymbol{x} \in \Omega \subset \mathbb{R}^d \\
u(\boldsymbol{x}, t) = 0 u(\boldsymbol{x}, 0) = f(\boldsymbol{x}) & \text{(initial condition)} \\
\frac{\partial}{\partial t} u(\boldsymbol{x}, 0) = g(\boldsymbol{x}) & \text{(initial condition)}
\end{cases}
$$

- general formulation of PDEs
- Ansatz: separation of variables
- Fundamental theorem on superposition
- Formulation of wave equation with boundary and initial conditions
- Solution formula for a wave equation on a bounded interval using separation of variables

8. Numerical Methods for Solving the Wave Equation

- Formulation of general two-point value problems
- Finite difference operators for first and second order derivatives and their approximation/convergence order
- Finite difference methods for general two-point value problem with Dirichlet boundary conditions
- Finite difference methods for general two-point value problem with Neuman/Robin boundary conditions
- d’Alembert’s solution formula for wave equation on real line

8. Heat Equation & Numerical Methods to solve it

$$
\begin{cases}
\frac{\partial}{\partial t}u - c^2 \frac{\partial^2}{\partial x^2}u = 0 & \\
u(0,t) = u(L,t) = 0 & \text{Dirichlet boundary conditions} \\
u(x,0) = f(x) & \text{initial conditions (at time } 0\text{)}
\end{cases}
$$

- Solution of the heat equation by Fourier series
- Steady two-dimensional heat equation: Laplace's equation
- Solving the Dirichlet problem in a rectangle
- Explicit & Implicit Euler's scheme.
- Crank-Nicolson
- Heat equation: Modelling very long bars & the Fourier Transform

9. Laplace Transform

$$F(s) = \mathcal{L}(f)(s) \int_0^\infty \mathrm{e}^{-st} f(t) \, \mathrm{d}t$$

- s-shifting & t-shifting
- existence & uniqueness of the Laplace transform
- Laplace transform of derivatives and integrals
- solving ODEs with the Laplace transform
- Heaviside (unit step) function u, Dirac's delta "function" δ
- Convolution & Integral Equations
- Systems of ODEs

10. Numerical Methods for Solving ODEs

- Initial value problem for ordinary differential equations (ODE)
- Euler's method and Heun's method,
  derivation of the methods and implementation
- Error theory including consistency/convergence error, convergence order
- Explicit Runge-Kutta methods: Motivation, description via Butcher tables, implementation
- Adaptive/embedded Runge-Kutta methods including computable error estimates and adaptive time-step selection
- Stability, the Stability function $R(z)$ & Stability region $\mathcal{S}$.
- Implicit Euler

The Exam.

May 15, 2023, 09:00 – 13:00

Allowed help material. Category C

- one sheet of A4 paper
  (“gul ark”)
  for own, handwritten notes.

  Fetch one at Sentralbygg II,
  Institutkontor IMF
  7. etasje.
- a simple calculator, see

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