10b Stepsize & Stability

13/04/23

10b-5, Consider

$f(t,y) = -2ty$

$$y' = -2ty \qquad y(0) = 1$$

known solution $y(t) = e^{-t^2}$ since $y'(t) = -2t e^{-t^2}$
$= -2t\, y(t)$

Compare Euler and Heun

$t_0 = 0$ $\qquad h = 0.1$

$$y_{n+1} = y_n + h f(t_n, y_n), \qquad y_0 = y(0) = 1$$

$$y_1 = y_0 + h f(t_0, y_0) = 1 + h(-2t_0 y_0)$$
$$= 1 + \underbrace{\boxed{0.1(-2\cdot 0\cdot 1)}}_{k_1} = 1$$

Heuns method $\qquad \hat{y}_0 = y(t_0) = 1 \qquad h = 0.1$

$$k_1 = f(t_0, y_0) = -2\cdot 0\cdot 1 \boxed{= 0}$$
$$k_2 = f(t_0 + h, y_0 + h k_1) = f(0.1, 1)$$
$$= -2(0.1)\cdot 1.0 = -0.2$$
$$\hat{y}_1 = \hat{y}_0 + \frac{h}{2}(k_1 + k_2)$$
$$= 1.0 + \frac{0.1}{2}(0 - 0.2) = 1.0 - \frac{0.02}{2} = \underline{\underline{0.99}}$$

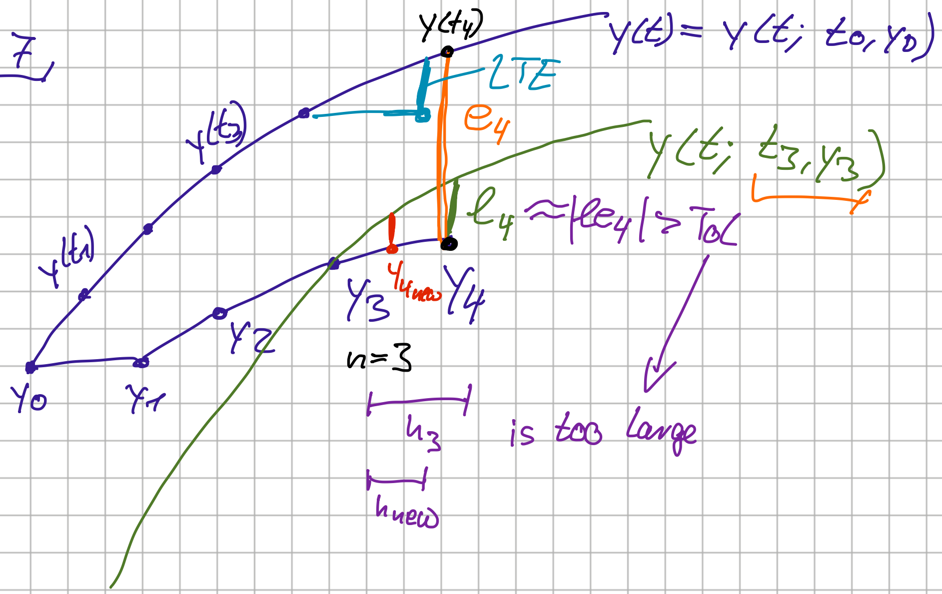
And we have

$$e_1 = y(t_1) - y_1 = e^{-0.1^2} - 1.0 \approx 0.995\cdot 10^{-2}$$

and

$$le_1 = \hat{y}_1 - y_1 = 0.99 - 1.0 \approx 0.1\cdot 10^{-2}$$

10-7,



105-11,

$$y' = \lambda y = \underline{f(t,y)} \quad , \quad y(t_0) = y(0) = y_0$$

$$y_{n+1} = y_n + h f(t_n, y_n) = y_n + h\lambda y_n = (1+h\lambda) y_n$$

$$= (1+h\lambda)^2 y_{n-1} = \cdots = \underbrace{(1+h\lambda)^n}\, y_0$$

for $t \to \infty \Rightarrow n \to \infty$ and we have a constant step size h

When does $y_{n+1} \to 0$? We need

$$\underline{|1+h\lambda| < 1}$$

for the method to have $y_n \to 0$ for $n \to \infty$

The step size h has to be chosen correctly!

Since $h > 0$, $\lambda < 0$ (for $\lambda \in \mathbb{R}$)

(1) $1 + h\lambda > 0 \Rightarrow 1 + h\lambda < 1 \Leftrightarrow \lambda h < 0$

(2) $1 + h\lambda < 0 \Rightarrow -1 - h\lambda < 1 \Leftrightarrow -2 < h\lambda$

$$\Rightarrow \quad -2 < h\lambda < 0$$

$$\Leftrightarrow \quad 0 < h|\lambda| < 2$$

$$\Leftrightarrow \quad 0 < h < \frac{2}{|\lambda|}$$

if $\lambda = -200 \Rightarrow$ We have to choose $h < \frac{1}{100}$

106-12 Heuns Method and $R(z)$

$$k_1 = f(t_n, y_n) = \lambda y_n$$

$$k_2 = f(t_n + h, y_n + h k_1) = \lambda(y_n + h k_1) = \lambda(y_n + h\lambda y_n)$$

$$y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2)$$

$$= y_n + \frac{h}{2}(\lambda y_n + \lambda y_n + \lambda^2 h y_n)$$

$$= (y_n + h\lambda y_n \quad \frac{(\lambda h)^2}{2} y_n)$$

$$= \left(1 + h\lambda + \frac{(h\lambda)^2}{2}\right) y_n$$

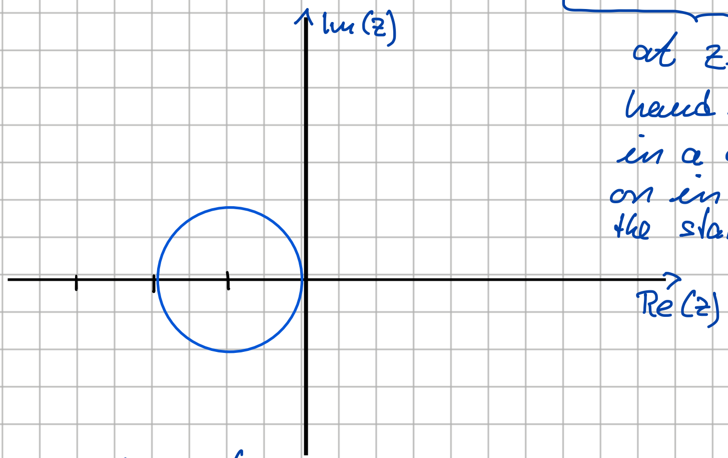
$$\Rightarrow R(z) = 1 + z + \frac{z^2}{2}$$

106-13

Euler: $R(z) = 1 + z$

$$\Rightarrow S = \{ z \in \mathbb{C} : \underbrace{|1+z| < 1}_{} \}$$

at $z=-1$ the left hand side is 0 and in a circle of radius on in $\mathbb{C}$ around that the statement holds



106-15 Stability function for implicit Euler

$$y_{n+1} = y_n + h f(t_{n+1}, y_{n+1}) = y_n + h\lambda y_{n+1}$$

$$\Leftrightarrow (1 - h\lambda) y_{n+1} = y_n$$

$$y_{n+1} = \frac{1}{1-h\lambda} y_n \quad \Rightarrow R(z) = \frac{1}{1-z}$$

$\lambda \in \mathbb{R}$, $\lambda < 0$. which h can I choose?

all of them!