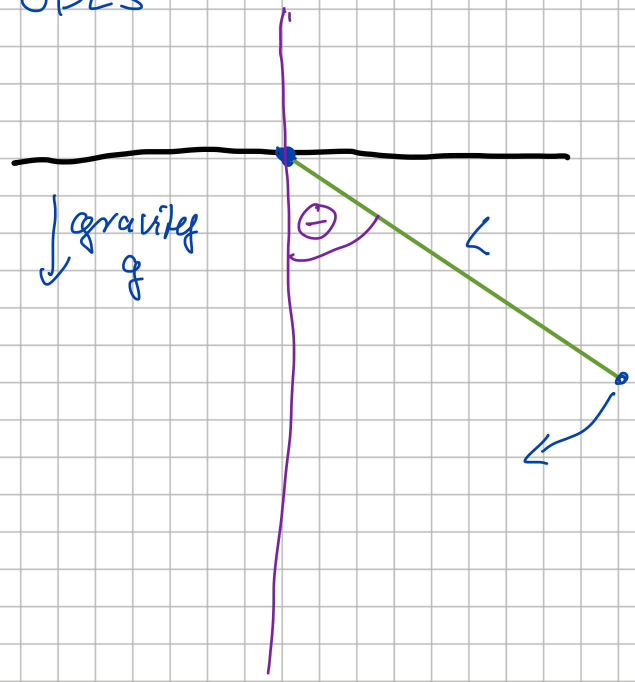
Numerics for ODEs 30/03/23

10-1, Pendulum



Goal. Where is the pendulum, when $\rho(t)$ measures Θ over time

and

$\rho(0) = \Theta_0$

$\rho'(0) = 0$

10-7, Introduce

$y_1(t) = u(t)$

$y_2(t) = u'(t)$

$y_3(t) = u''(t)$

$\vdots$

$y_m(t) = u^{(m-1)}(t)$

Consider

$y_1'(t) = (u(t))' = u'(t) = y_2(t)$

$y_2'(t) = (u'(t))' = u''(t) = y_3(t)$

$y_3'(t) = y_4(t)$

$y_m'(t) = (u^{(m-1)}(t))'$

$= u^{(m)}(t)$

$= f(t, u, u', \dots u^{(m-1)})$

$= f(t, y_1, y_2, \dots, y_m)$

10-12, Euler's Method

Idea 1, We Taylor our unknown function $y(t)$ at t_0 and evaluate at t_0+h

$$y(t_0+h) = y(t_0) + y'(t_0)h + \frac{1}{2}y''(t_0)h^2 + \dots$$

For small enough h, the higher order terms are small, so we ignore them

From the ODE we know

1) $y(t_0) = y_0$
2) $y'(t_0) = f(t_0, y(t_0)) = f(t_0, y_0)$

$$\Rightarrow \quad y(t_0+h) \approx y_0 + h f(t_0, y_0)$$

Since $t_0+h \approx t_1$

$$y_1 = y_0 + h f(t_0, y_0)$$

$$y_2 = y_1 + h f(t_1, y_1)$$

$$\vdots$$

$$y_{n+1} = y_n + h f(t_n, y_n)$$

Idea 2, We approximate with a forward difference

$$y'(t_0) \approx \frac{y(t_0+h) - y(t_0)}{h}$$

This is the same as

$$h f(t_0, y_0) \approx y(t_0+h) - \underbrace{y(t_0)}_{= y_0}$$

$$y(t_0+h) \approx y_1 = y_0 + h f(t_0, y_0)$$

10-13, Idea: Integrate!

$$\int_{t_n}^{t_n+h} y'(t)\, dt = \int_{t_n}^{t_n+h} f(t, y(t))\, dt$$

$$\Rightarrow \quad y(t)\Big|_{t=t_n}^{t_n+h} = \int_{t_n}^{t_{n+1}} f(t, y(t))\, dt$$

$$y(t_n+h) - y(t_n) = \int_{t_n}^{t_n+h} f(t, y(t))\, dt$$

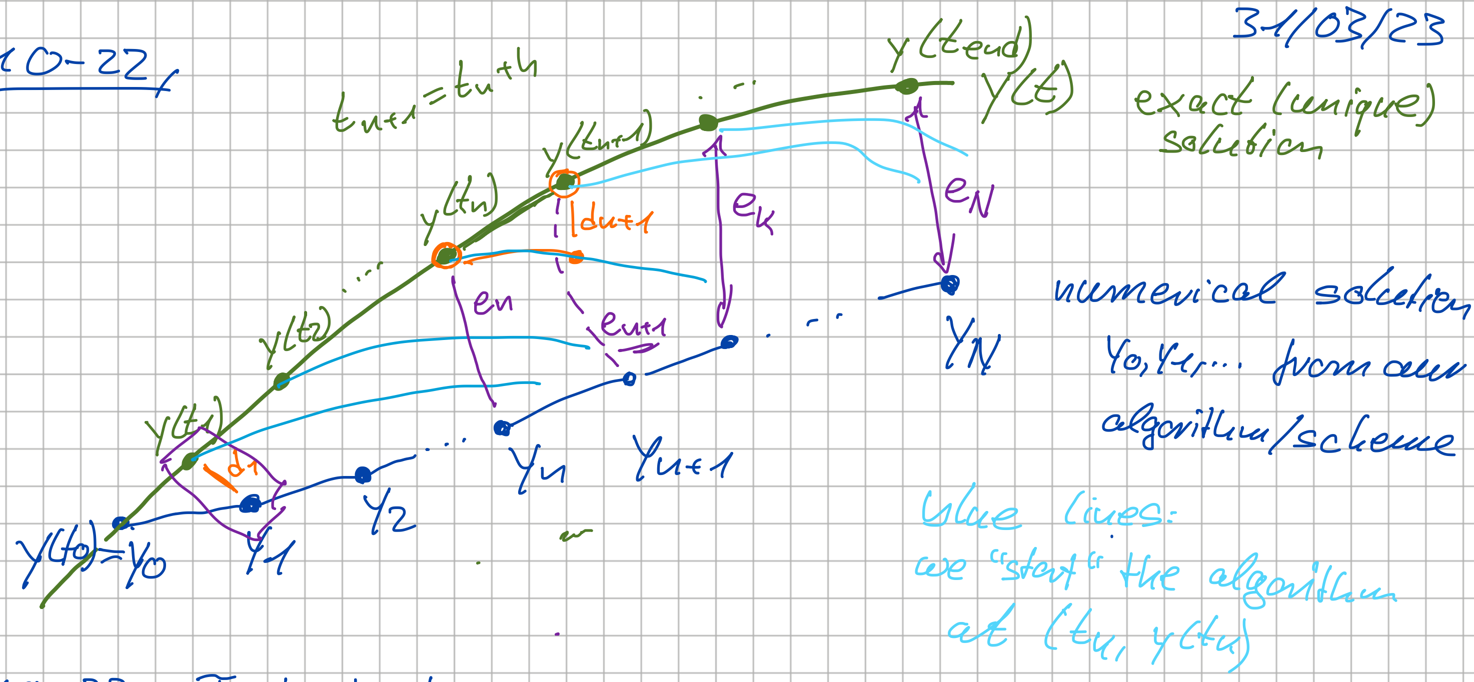
$$y(t_{n+1}) = y(t_n) + \underbrace{\int_{t_n}^{t_{n+1}} f(t, y(t))\, dt}_{\text{use trapezoidal rule}}$$

$$\Rightarrow \quad y(t_{n+1}) \approx y(t_n) + \frac{h}{2}\left[f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1})) \right]$$

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1}) \right]$$

10-22,

31/03/23



10-23, To find d_{n+1}

We have

a) $y(t)$ at $y(t_n)$ and $y(t_n+h)$

b) Euler's method starting at $(t_n, y(t_n))$

a) the Taylor expansion

$$y(t_n+h) = y(t_n) + h y'(t_n) + \frac{1}{2} y''(\xi)$$

$$t_n < \xi < t_{n+1}$$

b) One step starting from $(t_n, y(t_n))$

$$y(t_n+h) \approx y(t_n) + h \underbrace{f(t_n, y(t_n))}_{= y'(t_n)}$$

$$\underline{y(t_n+h)} \approx \underbrace{y(t_n) + h y'(t_n)}_{\text{it is the exact result of our numerical method}}$$

10-24, We use $$y(t_n+h) = y(t_n) + h f(t_n, y(t_n)) + d_{n+1}$$
$$y_{n+1} = y_n + h f(t_n, y_n)$$
$$e_n = y(t_n) - y_n$$

$$e_{n+1} = y(t_{n+1}) - y_{n+1}$$
$$= e_n + h\big(f(t_n, y(t_n)) - f(t_n, y_n)\big) + d_{n+1}$$

we use the mean value theorem

$$e_{n+1} = e_n + h f_y(t_n, \eta) \underbrace{(y(t_n) - y_n)}_{e_n} + d_{n+1},$$ η is between $y(t_n)$ and y_n

use triangle inequality

$$|e_{n+1}| \le |e_n| + h \underbrace{|f_y(t_n,\eta)|}_{\le L} \cdot |e_n| + \underbrace{|d_{n+1}|}_{\le \frac{1}{2}h^2|y''(\xi)| \le h^2 D}$$

we use

we get

$$|e_{n+1}| \le (1 + hL)|e_n| + Dh^2$$

- $|e_0| = y_0 - y(t_0) = 0$

$$|e_1| \le Dh^2$$
$$|e_2| \le (1+hL)Dh^2 + Dh^2 = ((1+hL)+1)Dh^2$$
$$|e_3| \le (1+hL)|e_2| + Dh^2 = ((1+hL)^2 + (1+hL) + 1)Dh^2$$
$$\vdots$$
$$|e_N| \le (1+hL)|e_{N-1}| + Dh^2 \le \sum_{n=0}^{N-1}(1+hL)^n Dh^2$$

We use the geometric sum

$$\sum_{n=0}^{N-1} r^n = \frac{r^N - 1}{r-1} \quad \text{for } r \in \mathbb{R}$$

and

$$e^x = 1 + x + \frac{1}{2}x^2 + \cdots = 1 + x + \sum_{n=2}^{\infty} \frac{x^n}{n!} \Rightarrow 1 + x < e^x$$ if $x > 0$

$$\sum_{n=0}^{N-1} \underbrace{(1+hL)^n}_{=r} = \frac{(1+hL)^N - 1}{1+hL-1} \leq \frac{(e^{hL})^N - 1}{hL}$$

$$= \frac{e^{hLN}-1}{hL} = \frac{e^{L(t_{end}-t_0)}-1}{hL}$$

$hN = t_{end} - t_0$

$$\Rightarrow |y(t_{end}) - y_N| = |e_N| \leq \underbrace{\frac{e^{L(t_{end}-t_0)}-1}{\cancel{h}L}}_{=C} Dh^2$$

$$\leq Ch$$

10-30,

Euler

$$\begin{array}{c|c} c_1 & a_{11} \\ \hline & b_1 \end{array} = \begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$$

$s = 1$

(1) $\sum_{i=1}^{s} b_i = 1$? $\qquad b_1 = 1$ ✓

(2) $\sum_{i=1}^{s} b_i c_i = \frac{1}{2}$? $\qquad b_1 c_1 = 0 \cdot 1 = 0 \neq 1/2$

not of order 2.

Heun

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array}$$

$s = 2$

(1) $\sum_{i=1}^{2} b_i = b_1 + b_2 = 1/2 + 1/2 = 1$ ✓

(2) $\sum_{i=1}^{2} b_i c_i = b_1 c_1 + b_2 c_2 = 0 \cdot 1/2 + 1 \cdot 1/2 = \frac{1}{2}$ ✓

(3) $\sum_{i=1}^{2} b_i c_i^2 = b_1 c_1^2 + b_2 c_2^2 = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1^2 = 1/2 \neq 1/3$

The method is of order 2