NTNU

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Laplace transform IV

Mathematics 4N

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Example MSD with shock load

MSD system with $M = 1$, $C = 4$ and $K = 3$ is resting.
At $t = 1$ we impose a "shock input".

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y''(t) + 4y'(t) + 3y(t) &= -\delta(t-1) \\
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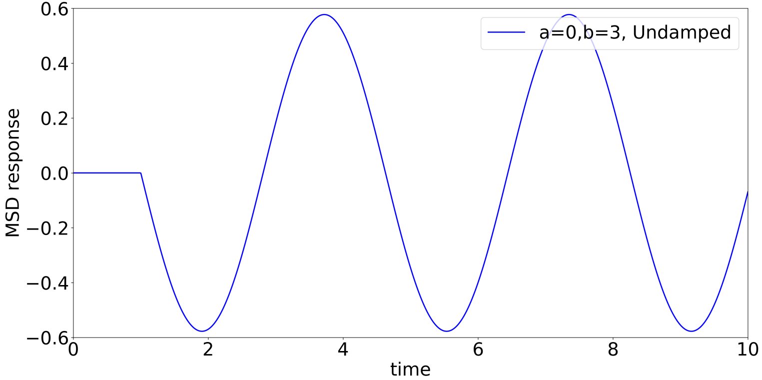
The result back in t reads:

$$y(t) = -\frac{1}{2}e^{-(t-1)}u(t-1) + \frac{1}{2}e^{-3(t-1)}u(t-1)$$

Example MSD with shock load



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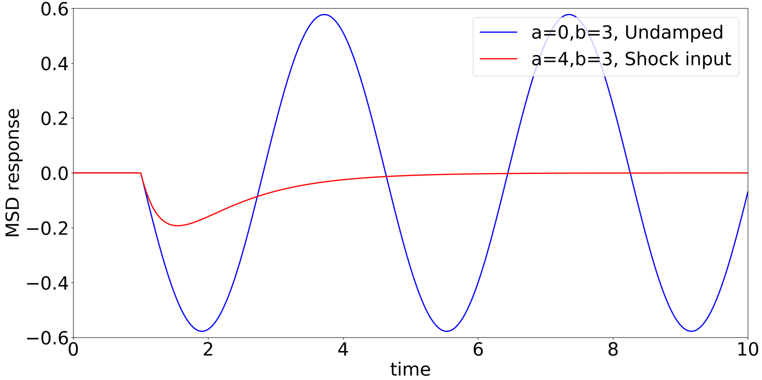

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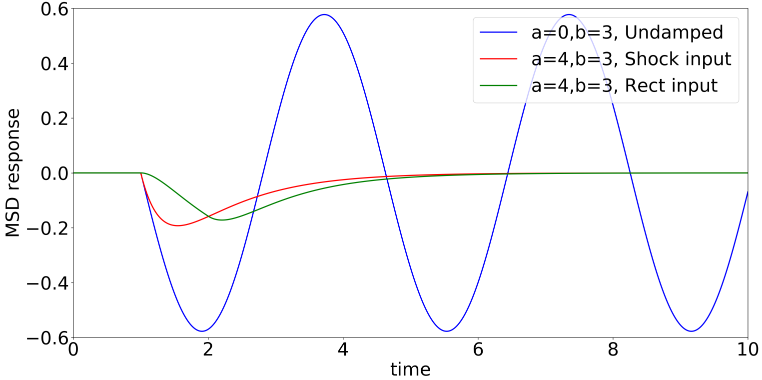

Outline

- Recap
- Motivation
- Convolution
- Integrals

Recap

We have seen rules/theorems on how to find the Laplace transform of:

- simple functions,
- linear combinations,
- multiplication with monomials,
- integrals of functions and
- derivatives of functions.

We have also seen two shifting theorems on how to find the LT and inverse.

Motivation

Reminder: The MSD dynamic system is characterised by its transfer function, $Q(s)$. In case of homogeneous ICs $Y(s) = Q(s)R(s)$.

What if we have complex inputs? Is there an easier way to find the inverse $y(t) = \mathcal{L}^{-1}(Y(s))$?

Similarly, for the RLC (RC) system, we solved it using the LT of integrals of functions.

Consider the case where we are coupling temperature change. Then the dielectric constant (or even the size of the conducting material) changes with temperature, which in turn changes in time, s.t. capacitance depends on time.

In case we have a complicated integro-differential equation, is there a way to more efficiently calculate the LT?

Convolution

It is stressed that while linearity states $\mathcal{L}(af(t)+bg(t)) = a\mathcal{L}(f(t)) + b\mathcal{L}(g(t))$, this does not hold for multiplication, i.e.

$$\mathcal{L}(f(t)g(t)) \neq \mathcal{L}(f(t))\mathcal{L}(g(t))$$

Instead, let us define convolution as follows:

Definition of convolution

Given two function $f(t)$ and $g(t)$, we define the convolution of f and g as:

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

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Convolution of $f(t) = e^{at}$ and $g(t) = 1$.

$$(e^{at} * 1)(t) = \int_0^t e^{a\tau} \cdot 1 d\tau = \frac{1}{a}\left(e^{at} - 1\right)$$

Laplace transform of convolution

Theorem

Given two function $f(t)$ and $g(t)$ with LT $\mathcal{L}(f)$ and $\mathcal{L}(g)$, the LT of their convolution reads:

$$\mathcal{L}\left((f * g)(t)\right) = \mathcal{L}(f(t))\mathcal{L}(g(t))$$

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Proof: (Hint) Change order of integration, then change of variable.

Properties of convolution

For the functions f, g, h, and assuming their convolution is defined we have the following properties:

- $f * g = g * f$
- $f * (g + h) = f * g + f * h$
- $f * (g * h) = (f * g) * h$
- $f * 0 = 0$

Convolution with Dirac's delta

The sifting property is reminded:

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$$

With convolution we have restricted bounds:

$$f(t) * \delta(t-a) = \int_{0}^{t} f(t-\tau)\delta(\tau-a)d\tau$$

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We consider cases:

$$f(t) * \delta(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(a) & \text{if } t \geq a \end{cases}$$

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Working with the MSD system we know that (in case of homogeneous IC):

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We need to find the inverse transform $\mathcal{L}^{-1}Q(s) = q(t)$.

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From convolution we can get a generalised expression:

$$y(t) = \frac{1}{2}e^{-t}\int_0^t r(\tau)e^{\tau}d\tau - \frac{1}{2}e^{-3t}\int_0^t f(\tau)e^{3\tau}d\tau$$

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Assume $r(t) = 1$:

$$y(t) = \frac{1}{3} - \frac{1}{2}e^{-t} + \frac{1}{6}e^{-3t}$$

Volterra integral equations

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$$Y(s) = \frac{R(s)}{1 - G(s)}$$

Volterra integral equations - Examples

Solve the Volterra equation when $g(t) = \sin(t)$ and $r(t) = t$.

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After factorisation and inverse transform we have the solution:

$$y(t) = t + \frac{1}{6}t^3$$