NTNU

Norwegian University of Science and Technology

Laplace transform III

Mathematics 4N

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Outline

- Motivation
- Heaviside function
- Second shifting theorem
- Dirac's delta function

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Motivation

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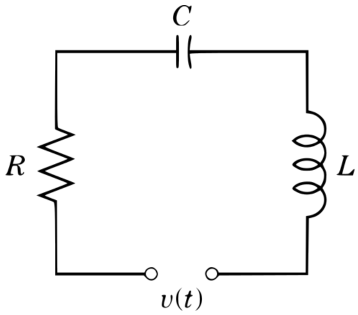
Last week we saw the mass-spring-damper (MSD) system as it moves towards zero-input equilibrium.

This week we will also consider an application of the resistor-inductor-capacitor (RLC) circuit.

The questions now become:

- What happens to the MSD system in case of shock loads?
- What is the response of an RLC circuit with an on-off switch?

The RLC circuit



$$Li'(t) + Ri(t) + \frac{1}{C}\int i(t)dt = v(t) \text{ for } t \geq 0,$$

where L is inductance, R is the resistance, C the capacitance, $i(t)$ the current and $v(t)$ the voltage input which is controlled by a **switch**.

Heaviside function

Definition

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The LT of the Heaviside function reads:

$$\mathcal{L}(u(t-a)) = \frac{e^{-as}}{s}$$

Rectangular functions

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Using linear combinations of the Heaviside function we can create rectangular functions:

$$r_a^b(t) = u(t-a) - u(t-b) \qquad a \leq t < b$$

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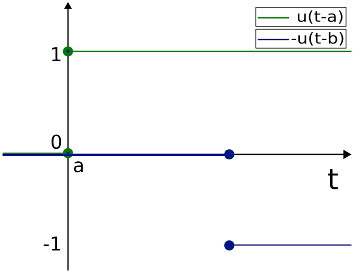

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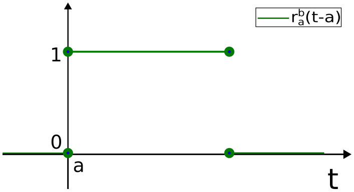

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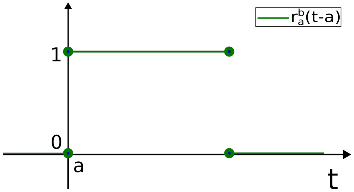
$$r_a^b(t) = u(t-a) - u(t-b) \qquad a \leq t < b$$

We can also introduce periodicity:

$$r_a(t) = u(t) - 2u(t-2a) + 2u(t-3a) - \dots$$



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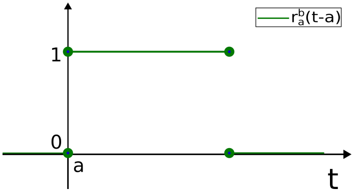
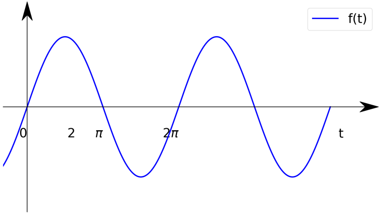
Formalised:

$$r_a(t) = u(t) + \sum_{i=1}^{\infty} (-1)^i 2u(t-(i+1)a)$$



Translated (and shifted) functions

Assume the signal input $f(t) = \sin(t)$



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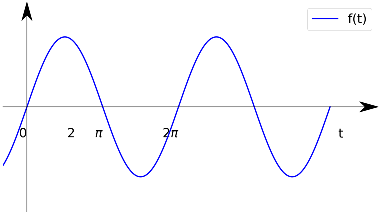

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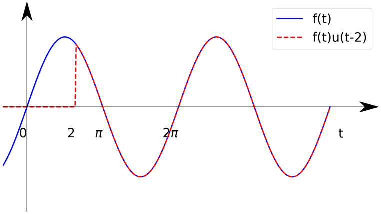
$$g(t) = u(t-a)f(t)$$



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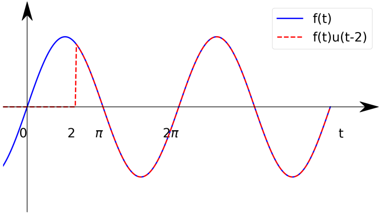

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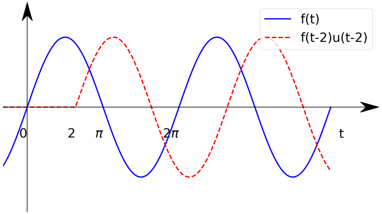

The t-shift theorem

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Theorem

If f has the LT $F(s)$. Then the "**shifted function**"

$$g(t) = f(t-a)u(t-a) = \begin{cases} 0 & t < a \\ f(t-a) & t \geq a \end{cases}$$

has the LT

$$\mathcal{L}(g(t)) = \mathcal{L}(f(t-a)u(t-a)) = e^{-as}F(s).$$

Equivalently, for the inverse:

$$f(t-a)u(t-a) = \mathcal{L}^{-1}(e^{-as}F(s)).$$

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Proof: (hint) Change of variable $\tau = t - a$.

Example

Find the LT of the function:

$$f(t) = \begin{cases} t & 0 \leq t < 1 \\ \sin(\frac{\pi t}{2}) & t \geq 1 \end{cases}.$$

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Can be written as:

$$f(t) = (u(t) - u(t-1))\, t + u(t-1)\sin(\frac{\pi t}{2}),$$

and the LT reads:

$$\mathcal{L}(f(t)) = \frac{1}{s^2} + e^{-s}\left(\frac{s+1}{s^2} + \frac{s}{s^2 + \frac{\pi^2}{4}}\right)$$

Example #2

We have an RC system with a switch (voltage input). The system is at rest until $t = a$, when we turn it on and at $t = b$ we turn if off again.

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The LT reads:

$$RI(s) + \frac{1}{C}\frac{1}{s}I(s) = v_0\left[\frac{e^{-as} - e^{-bs}}{s}\right],$$

or:

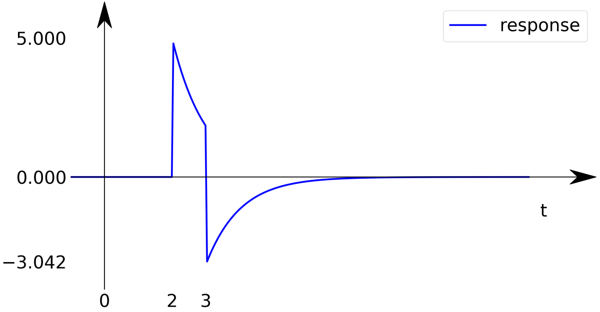
$$I(s) = \frac{v_0}{R}\left[\frac{e^{-as} - e^{-bs}}{s + \frac{1}{RC}}\right],$$

Example #2

The current response is:

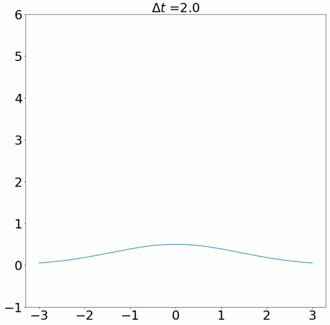
$$i(t) = \frac{v_0}{R}\left[u(t-a)e^{-\frac{1}{RC}(t-a)} - u(t-b)e^{-\frac{1}{RC}(t-b)}\right]$$

As an example, for $v_0 = 5$, $R = 1$, $C = 1$, $a = 2$ and $b = 3$ we get:



Dirac's delta function

Let us consider the MSD system. What happens if we have a shock load?



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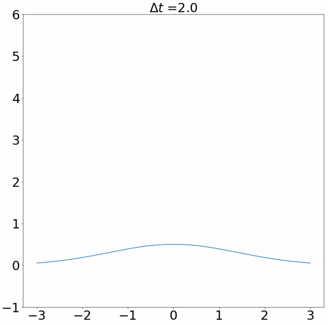

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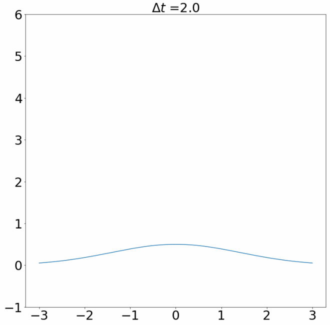
In an ideal setting, we can model such cases using Dirac's delta function. Let us start from the shifted function $f_k(t-a)$:

$$f_k(t-a) = \begin{cases} \frac{1}{k} & \text{if } a \leq t \leq a+k \\ 0 & \text{otherwise} \end{cases}.$$



Dirac's delta function & Properties

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Dirac's delta (**generalised**) function is the limit of $f_k(t-a)$ as $k \to 0$:

$$\delta(t-a) = \lim_{k \to 0} f_k(t-a) = \begin{cases} \infty & t = a \\ 0 & \text{otherwise} \end{cases}$$

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- $\int_{-\infty}^{\infty} \delta(t-a)f(t)dt = f(a)$
- Dirac's delta generalised function can be seen as the derivative of the Heaviside function.

Laplace transform of Dirac's delta function

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$$\mathcal{L}(\delta(t-a)) = e^{-as}$$

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Hint 2: Alternatively, use sifting property:

$$\int_{-\infty}^{\infty} \delta(t-a) e^{-st} dt = e^{-as}$$

Examples

MSD system with $M = 1$, $C = 0$ and $K = 1$ is resting.
At $t = 1$ we impose a "shock input".

$$\begin{aligned} y''(t) + y(t) &= -\delta(t-1), \text{ for } t \geq 0 \\ y'(0) &= 0 \\ y(0) &= 0 \end{aligned}$$

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Reminder:

$$Q(s) = \frac{1}{Ms^2 + Cs + K}$$

and

$$Y(s) = Q(s)\left[(Ms + C)y(0) + My'(0)\right] + R(s)Q(s)$$

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The result back in t reads:

$$y(t) = -u(t-1)\sin(t-1)$$