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Laplace transform II

Mathematics 4N

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Outline

- Recap
- Transform of derivatives
- Transform of integrals
- Solve ODEs

Recap

We have seen how to:

- check the existence of the Laplace transform,
- calculate the Laplace transform of functions,
- use linearity and known transformations to find the Laplace transform,
- find the inverse Laplace transform.

Laplace transform of derivatives

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Theorem

Consider a function $f\colon [0,\infty) \to \mathbb{R}$ (or $\mathbb{C}$), and let f' exist and be (piecewise) continuous in $[0,\infty)$. If the Laplace transform of f exists, then:

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0).$$

For the case that both f and f' are continuous, the Laplace transform of f' exists and the second derivative, f'', exists and is piecewise continuous, then:

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0).$$

Proof (hint): Integration by parts

Laplace transform of derivatives generalisation

Corollary

Let f, f',...,$f^{(n-1)}$ be continuous for all $t \geq 0$ and have "slower" than exponential growth, and let $f^{(n)}$ exist and be piecewise continuous. Then:

$$\mathcal{L}(f^{(n)}) = s^n \mathcal{L}(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0).$$

Note

We can rewrite the Laplace transform of the n^{th} derivative as follows:

$$\mathcal{L}(f^{(n)}) = s^n F(s) + p_{n-1}(s),$$

where $p_{n-1} \in \mathcal{P}^{n-1}$. Moreover, the polynomial coefficients, $f(0), f'(0), \ldots, f^{(n-1)}(0)$ are given by the initial conditions of the problem.

Example

Find the Laplace transform of $\sin(\omega t)$ using the derivative theorem.

Note

$$\cos(\omega t)' = -\omega \sin(\omega t)$$

and

$$\mathcal{L}\left(\cos(\omega t)\right) = \frac{s}{s^2 + \omega^2}$$

Multiplication theorem

Theorem

Let f be piecewise continuous. Then:

$$\mathcal{L}(tf(t)) = -\frac{d}{ds}\mathcal{L}(f) = -F'(s)$$

Proof: (hint) $te^{-st} = -\frac{d}{ds}e^{-st}$.

Example: Find the Laplace transform of $t\sin(t)$.

$$\mathcal{L}(t\sin t) = \frac{-2s}{(s^2+1)^2}$$

Laplace transform of integral of a function

Theorem

Let f be piecewise continuous for all $t \geq 0$ and have "slower" than exponential growth. Then for $s > 0$, $s > k$ and $t > 0$,

$$\mathcal{L}\left(\int_0^t f(\tau)d\tau\right) = \frac{1}{s}F(s).$$

Therefore,

$$\int_0^t f(\tau)d\tau = \mathcal{L}^{-1}\left(\frac{1}{s}F(s)\right).$$

Proof(hint): Show that $g(t) = \int_0^t f(\tau)d\tau$ fulfils the same growth criterion and use the Laplace transform of the derivative of a function.

Solving our first ODE

Let us consider a typical, mass-spring-damper system.

We have the 2nd order ODE:

$$\begin{cases} y''(t) + ay'(t) + by(t) = r(t) \\ y(0) = y_0 \\ y'(0) = u_0 \end{cases}$$

After the Laplace transform we get:

$$Y(s) = R(s)Q(s) + Q(s)\left((s+a)y_0 + u(0)\right),$$

where Q is the so-called transfer function that reads:

$$Q(s) = \frac{1}{s^2 + as + b}$$

Solving our first ODE

Some remarks:

- $r(t)$ is called input
- $y(t)$ is called output
- For homogeneous case, $Q(s) = \frac{Y(s)}{R(s)}$, hence its name.

Note $Q(s)$ depends only on a and b.

The solution by Laplace transform steps are reminded:

1. Find subsidiary function.
2. Solve the subsidiary equation using algebra.
3. Inverse transform solution back to the real variable.

Example 1 - Critically damped

Solve the ODE:

$$\begin{cases} y''(t) + 2y'(t) + y(t) = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

After the Laplace transform we get:

$$Y(s) = Q(s)(s+2),$$

with

$$Q(s) = \frac{1}{s^2 + 2s + 1}.$$

After the inverse transform we get the solution:

$$y(t) = (1+t)e^{-t}.$$

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Example 2 - Heavily damped

Solve the ODE:

$$\begin{cases} y''(t) + 4y'(t) + y(t) = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

After the Laplace transform we get:

$$Y(s) = Q(s)(s+2),$$

with

$$Q(s) = \frac{1}{s^2 + 2s + 1}.$$

We can rewrite $Y(s) = \frac{1/2 + 2/\sqrt{12}}{s+2-\sqrt{12}/2} + \frac{1/2 - 2/\sqrt{12}}{s+2+\sqrt{12}/2}$. Therefore,

$$y(t) = \left(\frac{1}{2} - \frac{2}{\sqrt{12}}\right) e^{-\left(2 - \frac{\sqrt{12}}{2}\right)t} + \left(\frac{1}{2} - \frac{2}{\sqrt{12}}\right) e^{-\left(2 + \frac{\sqrt{12}}{2}\right)t}.$$

Example 3 - Lightly damped

Solve the ODE:

$$\begin{cases} y''(t) + y'(t) + y(t) = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

After the Laplace transform we get:

$$Y(s) = Q(s)(s+1),$$

with

$$Q(s) = \frac{1}{s^2 + s + 1}.$$

We can rewrite $Y(s) = \frac{s+1/2}{(s+1/2)^2+3/4} + \frac{1/2}{(s+1/2)^2+3/4}$. Therefore,

$$y(t) = e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right).$$