NTNU
Norwegian University of Science and Technology

Laplace transformation I

Mathematics 4N

Vasileios Tsiolakis

March 16, 2023

Outline

- Motivation
- Introduction and definitions
- Properties
- Examples

Motivation

An integral transformation used to solve ordinary differential equations (ODEs).

Compared to the Fourier transformation, it is better suited for a lot of applications:

- Zero initial conditions,
- Decaying systems,
- Step functions.

Introduction

Let us consider some typical ODEs, i.e. find $y(t)$ s.t.

$$\begin{cases} y'(t) + ay(t) = r(t), \quad t > 0 \\ y(t_0) = K_0, \end{cases}$$

which is the case for exponential decay/growth,

or the kinematics equations:

$$\begin{cases} y''(t) + ay'(t) + by(t) = r(t), \quad t > 0 \\ y'(t_0) = K_0, \\ y(t_0) = K_1 \end{cases}$$

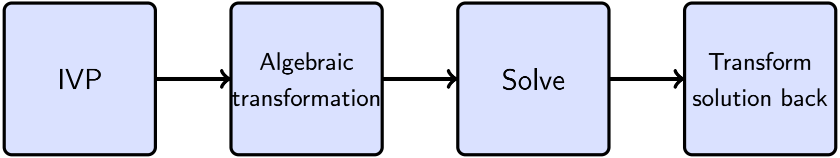
Recall from previous weeks, we could consider the Fourier transform, but it is far from efficient.

NTNU

Solving by Laplace transform

Instead using the Laplace transform, we follow the following concept:

1. Transform ODE of real variable to algebraic problem of complex variable.
2. Solve *subsidiary* equation using algebraic method.
3. Inverse transform solution back to the real variable.



Operators in shorthand

The Laplace transform is an operator similar to the rest.

Operators take a function f as input and output a new function.

Recall operators we have encountered.

- Derivative operator, D, which takes a differentiable function $f\colon [a,b] \to \mathbb{R}$ and returns a new function (its derivative) $D(f(x)) = f'(x)$.
- Integral operator, I, which takes an integrable function $f\colon [a,b] \to \mathbb{R}$ and returns a new function (its antiderivative) $I(f(t)) = \int_0^t f(t)dt$.
- Laplacian operator, Δ, which takes a twice-differentiable function $f\colon [a,b] \to \mathbb{R}$ and returns a new function $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f(x))$.

NTNU

Definition - The Laplace transform

Definition

Given a function $f(t)$ defined for $t \geq 0$, we define the Laplace transform as

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t) dt,$$

assuming that the integral exists.

That is typically the case for the applications considered.

The Laplace transform is an integral transformation with kernel e^{-st} and say it converges if the improper integral exists, i.e.

$$F(s) = \lim_{T \to \infty} \int_0^T e^{-st} f(t) dt.$$

We can similarly define the inverse transformation:

$$f(t) = \mathcal{L}^{-1}(F)$$

Notable examples

Let us have a look at some notable cases:

- $f(t) = c$, with $c \in \mathbb{R}$ and $t \geq 0$:

$$\mathcal{L}(c) = \int_0^\infty e^{-st} c dt = \frac{c}{s} \qquad \text{if } s > 0$$

- $f(t) = e^{at}$, $a \in \mathbb{R}$ and $t \geq 0$:

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at} dt = \frac{1}{s-a} \qquad \text{if } s > a$$

- $f(t) = e^{t^2}$, for $t \geq 0$:

$$\mathcal{L}(e^{t^2}) = \int_0^\infty e^{t^2 - st} dt \text{Does not exist}$$

Note that the Laplace transform is not defined for functions that grow faster than exponentials.

Existence of the Laplace transform

Conditions: Function must be piece-wise continuous **and** have slower than exponential growth.

Piecewise continuity

The function $f(t)$ defined in $[0, \infty)$ is continuous on any sub-interval $[t_i, t_{i+1}]$, with $0 = t_0 < t_1 < \cdots < t_n < \infty$ and its left and right limits ($\lim_{t \to t_i^+} f(t)$, $\lim_{t \to t_i^-} f(t)$) exist.

Growth

There exist numbers $M > 0$ and $a > 0$, such that:

$$|f(t)| \leq Me^{at}$$

If conditions are met, then $\mathcal{L}(f)$ is defined for any $s > a$.

Uniqueness of the Laplace transform

NTNU

Theorem

If $f(t)$ and $g(t)$ are defined and piecewise continuous in $[0, \infty)$ and their Laplace transform exists, then if

$$\mathcal{L}(f) = \mathcal{L}(g),$$

then we have that

$$f = g$$

on intervals that they are both continuous.

Linearity

Theorem

The Laplace transform is a linear operation; that is, for any functions $f(t)$ and $g(t)$ whose transforms exist and any constants a and b the transform of $af(t) + bg(t)$ exists, and

$$\mathcal{L}(af(t) + bg(t)) = a\mathcal{L}(f(t)) + b\mathcal{L}(g(t))$$

Laplace transform of cosine and sine

There are two ways to calculate them.

Reformulation into terms we have already seen:

Reminder

Euler's formula:

$$e^{ix} = \cos(\omega x) + i\sin(x)$$

- $f(t) = cos(\omega t)$, for $t \geq 0$:

$$\mathcal{L}(\cos(\omega t)) = \frac{s}{s^2 + \omega^2}$$

- $f(t) = sin(\omega t)$, for $t \geq 0$:

$$\mathcal{L}(\sin(\omega t)) = \frac{\omega}{s^2 + \omega^2}$$

NTNU

Laplace transform of cosine and sine

Integration by parts reads:

Reminder

$$\int_a^b f'(x)g(x)dx = f(x)g(x)|_a^b - \int_a^b f(x)g'(x)dx$$

We solve the following system to derive the Laplace transform for both functions:

$$\mathcal{L}(\cos(\omega t)) = \frac{1}{s} - \frac{\omega}{s}\mathcal{L}(\sin \omega t)$$
$$\mathcal{L}(\sin(\omega t)) = \frac{\omega}{s}\mathcal{L}(\cos \omega t)$$

Laplace transform of hyperbolic cosine and sine

Reformulation into terms we have already seen:

Reminder

$$\sinh(at) = \frac{1}{2}\left(e^{at} - e^{-at}\right)$$
$$\cosh(at) = \frac{1}{2}\left(e^{at} + e^{-at}\right)$$

We use the linearity to get:

$$\mathcal{L}(\sinh(at)) = \frac{1}{2}\frac{1}{s-a} - \frac{1}{2}\frac{1}{s+a}$$
$$\mathcal{L}(\cosh(at)) = \frac{1}{2}\frac{1}{s-a} + \frac{1}{2}\frac{1}{s+a}$$

NTNU

Laplace transform of monomials

Let us consider the function $f(t) = t$, for $t \geq 0$. Once again we will use integration by parts:

Then the Laplace transform reads:

$$\mathcal{L}(t) = \frac{1}{s^2}$$

Let us generalise the formulation with $f(t) = t^n$, for $t \geq 0$:

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

Laplace transform of monomials

NTNU

Let us consider the function $f(t) = t$, for $t \geq 0$. Once again we will use integration by parts:

Then the Laplace transform reads:

$$\mathcal{L}(t) = \frac{1}{s^2}$$

Let us generalise the formulation with $f(t) = t^n$, for $t \geq 0$:

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

Table of Laplace transforms

Concept: Instead of trying to calculate the Laplace transform, use linearity and known transforms.

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$

$f(t)$	$F(s)$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sinh(at)$	$\frac{a}{s^2+a^2}$
$\cosh(at)$	$\frac{s}{s^2+a^2}$

Example

Concept: Instead of trying to calculate the Laplace transform, use linearity and known transforms.

Compute the Laplace transform of $f(t) = t^4 + e^t + 3\cos(2t)$

$$F(s) = \frac{4!}{s^5} + \frac{1}{s-1} + 3\frac{s}{s^2+4}$$

First shifting theorem

s-shifting theorem

If $f(t)$ has a Laplace transform $F(s)$ (for all $s > k$, for some k), then $e^{at}f(t)$ has the transform $F(s-a)$ (for all $s - a > k$).

$$\mathcal{L}(e^{at}f(t)) = F(s-a)$$

Inverse transform example

We combine the table of known transforms and the shifting theorem to find the inverse.

Example 1: Find the function $f(t)$ if

$$\mathcal{L}(f) = \frac{3s - 137}{s^2 + 2s + 401}$$

$$f(t) = e^{-t}(3\cos 20t - 7\sin 20t)$$

Example 2: Find the function $f(t)$ if

$$\mathcal{L}(f) = \frac{17 - 2s}{s^2 - 3s - 10}$$

$$f(t) = e^{5t}e^{-2t}$$