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Numerics for the wave equation

Mathematics 4N

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Outline

- Introduction/Motivation
- The wave equation
- Finite differences

(Brief) introduction/motivation

A wave is a propagating dynamic disturbance.

Waves are of particular interest:

- Seismic waves, oscillating stresses,
- Surface waves, aeroacoustics,
- Particle movement/excitation,
- etc.

Great entry-point to numerical methods, great transition to non-linear problems.

The wave equation

The wave equation reads:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c\frac{\partial^2 u}{\partial x}, && \text{for } t > 0, x \in (0, L), \\ u &= 0, && \text{for } t > 0, x = 0, L, \\ u &= f(x), && \text{for } t = 0, x \in (0, L), \\ \frac{\partial u}{\partial t} &= g(x) && \text{for } t = 0, x \in (0, L), \end{aligned}$$

where $u(x, t)$ is the position of the wave at this space and time and c is the wave velocity and has been fixed.

Finite Differences

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Considering a discretised computational domain with characteristic sizes h and Δt in space and time, respectively, we can write the central differencing (CD) schemes:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} + \mathcal{O}(h^2)$$

$$\frac{\partial^2 u}{\partial t^2} \approx \frac{u(x,t+\Delta t) - 2u(x,t) + u(x,t-\Delta t)}{\Delta t^2} + \mathcal{O}(\Delta t^2)$$

Finite difference scheme of the wave equation

Transferring the schemes in the wave equation we get:

$$U_i^{n+1} = -U_i^{n-1} + 2\left(1 - \frac{c^2 \Delta t^2}{\Delta x^2}\right) U_i^n + \frac{c^2 \Delta t^2}{\Delta x^2}\left(U_{i-1}^n + U_{i+1}^n\right), \tag{1}$$

where i notes the position in space (x) and n the position in time (t). In this equation U_i^{n+1} is the unknown at position $x_i = x_o + i\Delta x$ and time-step $t_{n+1} = t_0 + (n+1)\Delta t$.

If $\alpha = \frac{c\Delta t}{\Delta x}$ we can rewrite (1) as follows:

$$U_i^{n+1} = -U_i^{n-1} + 2\left(1 - \alpha^2\right) U_i^n + \alpha^2\left(U_{i-1}^n + U_{i+1}^n\right), \tag{2}$$

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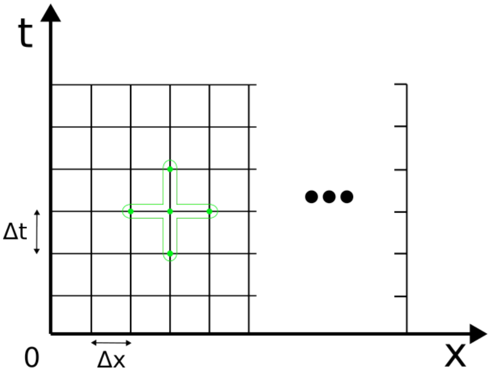
Finite difference scheme of the wave equation



With green we outline the so-called stencil, i.e. a visual representation of the known values required to compute the unknown. The unknown, U_i^{n+1}, is top-most point of the stencil. Based on the stencil, we can make a series of observations.

Remarks on explicit/implicit schemes

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In the scheme described above, we formulate an equation starting from time-step t_1, or $n = 0$. From (2) and for any i, we see that all terms required for the calculation of our U_i^1 are known (more on that later). Such schemes are called explicit. On the contrary, if each unknown U_i^{n+1} is dependent on all other unknowns, it is called an implicit scheme.

Therefore, for explicit schemes, for any time-step t_{n+1}, we can loop through all x_i positions and calculate u_i^{n+1} before moving to time-step t_{n+2} (outer loop). Such algorithm is called **explicit time-marching**.

Remarks on $\alpha = \frac{c\Delta t}{\Delta x}$

The number $\alpha = \frac{c\Delta t}{\Delta x}$ is called the Courant number (might also find it as c in literature with the convective velocity being a instead of c).

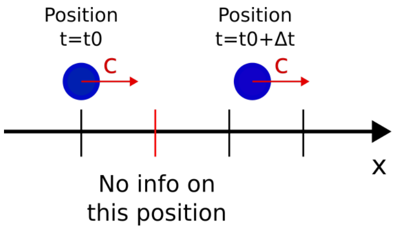
The Courant-Friedrichs-Lewy (CFL) condition for explicit time-marching schemes states that (in our case):

$$\alpha = \frac{c\Delta t}{\Delta x} \leq \alpha_{crit} = 1.$$

In other words, for a given spatial resolution, there exists a maximum allowable time-step Δt.

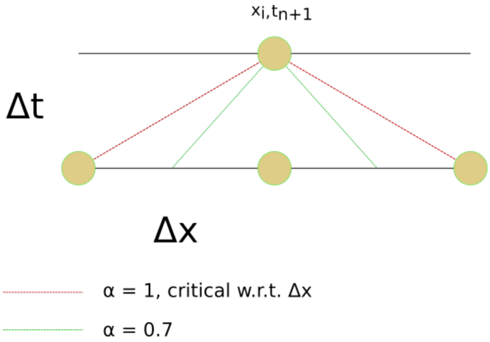
Remarks on $\alpha = \frac{c\Delta t}{\Delta x}$

From a rather practical point of view, in order for a model (using our stencil) to describe the characteristics of a body, moving with speed c, at every position in space, the body cannot move more than Δx over a time-interval Δt. Otherwise, we do not have a way to derive information for intermediate positions.



Remarks on $\alpha = \frac{c\Delta t}{\Delta x}$

Combining (2) and our mesh, we can see that for $\alpha = 1$, the solution depends only on the left and right neighbours. For $\alpha < 1$, it depends on all 3 neighbours, whereas for $\alpha > 1$ it becomes unphysical.



Remarks on stencil centred around x_i, t_0

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For $n = 0$, namely, the unknown is U_i^1, we require knowledge of the time-step $t = t_0 - \Delta t$. These "non-physical" points are called phantom nodes. We derive information about them from the initial conditions. There are temporal discretisation schemes that do not require phantom nodes.

From the initial condition $\frac{\partial u}{\partial t}|_{t=0} = g(x)$ we get:

$$U_i^{-1} = U_i^1 - 2\Delta t g(x_i) \tag{3}$$

Combining (2), (3) and the other IC we get:

$$U_i^1 = \left(1 - \alpha^2\right) f(x_i) + \frac{\alpha^2}{2} \left(f(x_{i-1}) + f(x_{i+1})\right) + \Delta t g(x_i). \tag{4}$$

Example

$\Delta t = 1$, $\Delta x = 1$, $c = 1$, $L = 3$, $g(x) = 0$, $f(x) = -x(x-3)$ and $u(x=0) = 0$, $u(x=L) = 0$.

The Courant number is $\alpha = 1$.

From (4) we have:

$$U_i^1 = \frac{1}{2}\left(f(x_{i-1}) + f(x_{i+1})\right)$$
$$U_1^1 = 0.5(0+2) = 1$$
$$U_2^1 = 0.5(2+0) = 1$$

For every subsequent time-step we have:

$$U_i^{n+1} = -U_i^{n-1} + \left(U_{i-1}^n + U_{i+1}^n\right)$$
$$U_1^2 = -2 + (0+1) = -1$$
$$U_2^2 = -2 + (1+0) = -1$$

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