NTNU

Norwegian University of Science and Technology

D'Alembert's solution
The heat equation

Mathematics 4N

Vasileios Tsiolakis

March 3, 2023

Outline

- Recap:
  - Continuing with D'Alembert's solution of the wave equation
- The heat equation
  - Separation of variables
  - Steady 2D heat equation

Where we left off - D'Alembert's solution

Starting from the wave equation:

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x} && \text{for } t > 0,\ x \in (0, L), \\
u &= 0 && \text{for } t > 0,\ x = 0, L, \\
u &= f(x) && \text{for } t = 0,\ x \in (0, L), \\
\frac{\partial u}{\partial t} &= g(x) && \text{for } t = 0,\ x \in (0, L),
\end{aligned}
\tag{1}
$$

we introduce a change of variables

$$\xi = x + ct, \qquad \eta = x - ct$$

D'Alembert's solution of the wave equation

Solution is $u(\xi, \eta)$. We want to express (1) in terms of ξ, η.

Chain rule:

$$\begin{aligned}\frac{\partial}{\partial x} u(\xi, \eta) &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} \\ &= \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\end{aligned} \tag{2}$$

Applying the chain rule to the RHS of (2):

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} u(\xi, \eta)\right) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) \\ &= \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \eta^2}\end{aligned} \tag{3}$$

D'Alembert's solution of the wave equation

Same process for the temporal term.

Chain rule:

$$\begin{aligned}\frac{\partial}{\partial t}u(\xi,\eta) &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} \\ &= c\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)\end{aligned} \tag{4}$$

Applying the chain rule to the RHS of (4)

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}u(\xi,\eta)\right) &= \frac{\partial}{\partial t}\left[c\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)\right] \\ &= c^2\left(\frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \eta^2}\right)\end{aligned} \tag{5}$$

NTNU

D'Alembert's solution of the wave equation

Transferring (3) and (5) to (1):

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \tag{6}$$

A solution of the form

$$u = \phi(\xi) + \psi(\eta)$$

or in terms of x and t

$$u = \phi(x + ct) + \psi(x - ct)$$

is known as D'Alembert's solution of the wave equation.

D'Alembert's solution of the wave equation

Finding the functions ϕ and ψ:

Let us have a look at the initial conditions.

$$\vdots$$

D'Alembert's solution of the wave equation

Finding the functions ϕ and ψ:

Let us have a look at the initial conditions.

$$\vdots$$

$$\phi(x+ct)+\psi(x-ct)=\frac{1}{2}\left[f(x+ct)+f(x-ct)\right]+\int_{x-ct}^{x+ct} g(s)ds$$

We can draw parallels from the solution with separation of variables.

Derivation of the heat equation

Some notational work and assumptions:

- Let σ be the specific heat and ρ the density of the material of a body. Isotropic body.
- Heat flows in the direction of decreasing temperature and is proportional to the temperature gradient, i.e. $\boldsymbol{v} = -K\boldsymbol{\nabla} u$.
- Thermal conductivity K is constant (homogeneous material).

The heat equation

Dirichlet

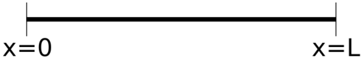
$$
\begin{aligned}
\frac{\partial u}{\partial t} &= c^2 \boldsymbol{\nabla}^2 u && \text{in } \Omega \times t, t > 0 \\
u(\boldsymbol{x}, t) &= 0 && \text{on } \Gamma \times t \\
u(\boldsymbol{x}, 0) &= f(\boldsymbol{x}) && \text{in } \Omega \text{ at } t = 0.
\end{aligned} \tag{7}
$$

Neumann

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= c^2 \boldsymbol{\nabla}^2 u && \text{in } \Omega \times t, t > 0 \\
\boldsymbol{\nabla}_n u &= 0 && \text{on } \Gamma \times t \\
u(\boldsymbol{x}, 0) &= f(\boldsymbol{x}) && \text{in } \Omega \text{ at } t = 0.
\end{aligned} \tag{8}
$$

Heat equation (Homogeneous Dirichlet, 1D)

Consider a cable of length L, with temperature $u = 0$ at each end.



$$
\begin{aligned}
\frac{\partial u}{\partial t} &= c^2 \frac{\partial^2 u}{\partial x^2} && x \in [0, L], t > 0, \\
u &= 0 && \text{for } x = 0, L \text{ and } t > 0 \\
u &= f(\boldsymbol{x}) && \text{for } x \in (0, L) \text{ at } t = 0.
\end{aligned}
$$

Heat equation (1D)

Assume separable solution field $u(x,t) = F(x)G(t)$

$$F\dot{G} = c^2 F^{''} G$$

and divide by c^2FG to get:

$$\frac{\dot{G}}{c^2 G} = \frac{F^{''}}{F} = -p^2$$

Multiplying by the denominators we get 2 conditions:

$$F^{''} + p^2 F = 0$$
$$\dot{G} + p^2 c^2 G = 0$$

Heat equation (1D)

Starting from $F'' + p^2 F = 0$ and enforcing the BCs:

Heat equation (1D)

Starting from $F'' + p^2 F = 0$ and enforcing the BCs:

$$F_n(x) = \sin \frac{n\pi x}{L}, \qquad n = 1, 2, \ldots$$

*Note: In case of Neumann BCs:

$$F_n(x) = \cos \frac{n\pi x}{L}, \qquad n = 1, 2, \ldots$$

NTNU

Heat equation (1D)

Starting from $F'' + p^2 F = 0$ and enforcing the BCs:

$$F_n(x) = \sin\frac{n\pi x}{L}, \qquad n = 1, 2, \ldots$$

*Note: In case of Neumann BCs:

$$F_n(x) = \cos\frac{n\pi x}{L}, \qquad n = 1, 2, \ldots$$

Solving $\dot{G} + p^2 c^2 G = 0$ for $p = n\pi/L$:

Heat equation (1D)

Starting from $F'' + p^2 F = 0$ and enforcing the BCs:

$$F_n(x) = \sin \frac{n\pi x}{L}, \qquad n = 1, 2, \ldots$$

*Note: In case of Neumann BCs:

$$F_n(x) = \cos \frac{n\pi x}{L}, \qquad n = 1, 2, \ldots$$

Solving $\dot{G} + p^2 c^2 G = 0$ for $p = n\pi/L$:

$$G_n(t) = B_n e^{-\lambda_n^2 t}, \qquad n = 1, 2, \ldots$$

where $\lambda_n = cn\pi/L$.

Heat equation (1D)

Putting everything together:

$$u_n(x,t) = F_n(x)G_n(t) = B_n \sin\frac{n\pi x}{L} e^{-\lambda_n^2 t}, \qquad n = 1, 2, \ldots$$

eigenfunctions of the problem, for **eigenvalues** $\lambda_n = cn\pi/L$

*Family of solutions of the heat equation, fulfilling the boundary conditions.

Heat equation (1D)

Putting everything together:

$$u_n(x,t) = F_n(x)G_n(t) = B_n \sin\frac{n\pi x}{L} e^{-\lambda_n^2 t}, \qquad n = 1, 2, \dots$$

eigenfunctions of the problem, for **eigenvalues** $\lambda_n = cn\pi/L$

*Family of solutions of the heat equation, fulfilling the boundary conditions.

Using a linear combination of all possible solutions and enforcing the IC we evaluate B_n

Heat equation (1D)

Putting everything together:

$$u_n(x,t) = F_n(x)G_n(t) = B_n \sin\frac{n\pi x}{L} e^{-\lambda_n^2 t}, \qquad n = 1, 2, \ldots$$

eigenfunctions of the problem, for **eigenvalues** $\lambda_n = cn\pi/L$

*Family of solutions of the heat equation, fulfilling the boundary conditions.

Using a linear combination of all possible solutions and enforcing the IC we evaluate B_n

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

**or the coefficients of the odd Fourier functions. In case of Neumann BCs, we are looking for coefficients of the even Fourier series $(a_0, a_1, \ldots, a_n)$.

Steady 2D heat equation

The heat equation in 2 dimensions reads:

$$\frac{\partial u}{\partial t} = c^2 \boldsymbol{\nabla}^2 u = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

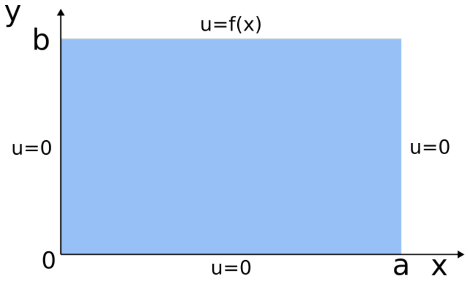
Steady 2D heat equation

The heat equation in 2 dimensions reads:

$$\frac{\partial u}{\partial t} = c^2 \nabla^2 u = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

However for the steady case $\frac{\partial u}{\partial t} = 0$.

We consider the following Dirichlet conditions.



Steady 2D heat equation

This time we are considering a separable expression in the form:

$$u(x, y) = F(x)G(y)$$

Much like before we end up with two equations:

$$\frac{\partial^2 F}{\partial x^2} + kF = 0$$
$$\frac{\partial^2 G}{\partial y^2} - kG = 0$$

Steady 2D heat equation

This time we are considering a separable expression in the form:

$$u(x, y) = F(x)G(y)$$

Much like before we end up with two equations:

$$\frac{\partial^2 F}{\partial x^2} + kF = 0$$
$$\frac{\partial^2 G}{\partial y^2} - kG = 0$$

Enforcing left and right BCs (along x axis) we get:

$$F(x) = F_n(x) = \sin \frac{n\pi}{a} x$$

and $k = (n\pi/a)^2$

Steady 2D heat equation

This time we are considering a separable expression in the form:

$$u(x,y) = F(x)G(y)$$

Much like before we end up with two equations:

$$\frac{\partial^2 F}{\partial x^2} + kF = 0$$
$$\frac{\partial^2 G}{\partial y^2} - kG = 0$$

Enforcing left and right BCs (along x axis) we get:

$$F(x) = F_n(x) = \sin \frac{n\pi}{a} x$$

and $k = (n\pi/a)^2$

We plug k in the second equation and :

$$G(y) = G_n(y) = A_n(e^{n\pi y/a} - e^{-n\pi y/a}) = 2A_n \sinh \frac{n\pi y}{a}$$

Steady 2D heat equation

NTNU

If $2A_n = A_n^\star$, we obtain the eigenfunctions

$$u_n(x,y) = F_n(x)G_n(y) = A_n^\star \sin\frac{n\pi}{a}x \sinh\frac{n\pi y}{a}$$

Steady 2D heat equation

If $2A_n = A_n^\star$, we obtain the eigenfunctions

$$u_n(x,y) = F_n(x)G_n(y) = A_n^\star \sin \frac{n\pi}{a} x \sinh \frac{n\pi y}{a}$$

Summing all eigenfunctions and enforcing the non-zero condition we get:

$$b_n = A_n^\star \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) sin \frac{n\pi x}{a} dx.$$

Therefore,

$$A_n^\star = \frac{2}{a \sinh n\pi b/a} \int_0^a f(x) sin \frac{n\pi x}{a} dx.$$

Compare to the 1-dimensional case.