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TMA4125 Matematikk 4N

Partial Differential Equations.

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Partial Differential Equations

Definition. A partial differential equation (PDE) is an equation that involves one or more partial derivatives of an (unknown) function u with at least two independent variables (multivariate).

We often use $u(x,t)$, $u(x,y,t)$ or $u(x,y,z,t)$ for a function that depends on space (1D, 2D, or 3D, respectively) and time (t).

- the PDE is linear if it is of first degree in u and its derivatives.
- otherwise it is called nonlinear.
- It is called homogenous if all terms include u or one of its partial derivatives
- otherwise it is called nonhomogeneous
- The order of a PDE is the order of the highest partial derivative appearing in the PDE

Specifying these is called Classification of the PDE

Important (one-dimensional) Examples

The (1D) heat equation. Given some heat source $q(x,t)$ and some α, find the temperature $u(x,t)$ which fulfils

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = q(x,t)$$

The (1D) wave equation. Find u that fulfils

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

where c is the wave speed.

Vibration of an elastic beam (bar). Let $q(x,t)$ denote some mechanical load of the bar. Find u which fulfils

$$\frac{\partial^2 u}{\partial t^2} + k^2 \frac{\partial^4 u}{\partial x^4} = q(x,t),$$

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Important (2D/3D) Examples

The (2D) wave equation. Find $u = u(t, x, y)$ that fulfils

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

where c is the wave speed

The (2D) Laplace equation. Find $u = u(t, x, y)$ that fulfils

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Introducing $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ we can write this is in short as

$$\Delta u = 0$$

The (2D) Poisson equation. given some $f = f(x, y)$, find $u(x, y)$ such that

$$\Delta u = f.$$

Solutions of partial Differential Equations

A solution u of a PDE in some region $\Omega \subset D$ in the space of its variables t, x (y,z) is a function whose partial derivatives (appearing in the PDE) exist in D and such that u fulfils the PDE on Ω.

The set of solutions might be huge, so for a unique solution we additionally require for example

- that u is given on the boundary of the region Ω
  these are called boundary condtions
- that u has some conditions for the start time $t = 0$
  these are called initial conditions

Note. For a PDE of order k er need k initial conditions

Superposition principle

Theorem. (Superposition principle) If u_1 and u_2 are solutions of a homogeneous linear PDE on some Ω, then

$$u = au_1 + bu_2, \qquad \text{for some constants } a, b$$

is also a solution of that PDE in the region Ω.

Note. This also means $u \equiv 0$ is always a solution to a homogeneous linear PDE.

Derivation of the Wave equation

Model.
A one-dimensional vibrating string on $[0, L]$
that is fixed on the ends, i.e. $u(t, 0) = u(t, L) = 0$ for all time t.

Goal.
The function $u(t, x)$ should describe the vertical position of our string at time t and position $x \in [0, L]$.

Assumptions.

1. uniform mass (“homogeneous string”)
2. we neglect gravity, i.e. only stretch and tension forces act on our string
3. at every point the string only moves vertical

The Wave Equation

the one-dimensional wave equation is the PDE for some $L, c > 0$

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} u = c^2 \dfrac{\partial^2}{\partial x^2} u & x \in [0, L] \\ u(0,t) = u(L,t) = 0 & \mathbf{x} \in \partial\Omega \quad \text{(boundary conditions)} \\ u(x,0) = f(x) & \text{(initial condition)} \\ \dfrac{\partial}{\partial t} u(x,0) = g(x) & \text{(initial condition)} \end{cases}$$

Note. For the wave equation we need two initial conditions:

- The initial position $f(x)$
- The initial velocity $g(x)$

Example (Desmos)

The Wave Equation

the multi-dimensional wave equation is the PDE for some $\Omega, c > 0$

$$
\begin{cases}
\frac{\partial^2}{\partial t^2} u = c^2 \Delta u & \mathbf{x} \in \Omega \subset \mathbb{R}^d \\
u(\mathbf{x}, t) = 0 & \mathbf{x} \in \partial\Omega \quad \text{(boundary conditions)} \\
u(\mathbf{x}, 0) = f(\mathbf{x}) & \text{(initial condition)} \\
\frac{\partial}{\partial t} u(\mathbf{x}, 0) = g(\mathbf{x}) & \text{(initial condition)}
\end{cases}
$$

Note. For the wave equation we need two initial conditions:

- The initial position $f(\mathbf{x})$
- The initial velocity $g(\mathbf{x})$

Example (Desmos)

Ansatz: Separation of variables

Ansatz. (or Idea: What if our) solution can be written as

$$u(x,t) = F(x)G(t)$$

We obtain

$$\frac{G''(t)}{c^2 G(t)} = -k = \frac{F''(x)}{F(x)} \qquad \text{for some constant } k \in \mathbb{R}$$

(we choose $-k$ just such that the following derivations are nicer)

or in other words two ordinary differential equations (ODEs)

$$F''(x) - kF(x) = 0$$
$$G''(t) - c^2 kG(t) = 0$$

Since k is some constant, let's take a look at different cases of k next.

Separation of Variables, Case $k = 0$ in the two equations.

Short summary of handwritten notes. Since $F''(x) = 0$ we have $F(x) = Ax + B$.
The boundary conditions $u(0,t) = u(L,t) = 0$ yield either $F(x) = 0$ or $G(t) = 0$, so in both cases

$$u(x,t) = 0 \quad \text{for all } x, t,$$

which is not an interesting solution.

Case $k > 0$ in $F''(x) - kF(x) = 0$

Short summary of handwritten notes. We have to solve a linear system starting from the linear combination of the fundamental solutions, but we also obtain $A = B = 0$ or $F(x) = 0$, so

$$u(x,t) = 0 \quad \text{for all } x, t,$$

which is (again) not an interesting solution.

Case $k < 0$ in $F''(x) - kF(x) = 0$

Short summary of handwritten notes. We first obtain that for for $k = -\frac{n\pi}{L}$, $n = 1, 2, ...$ a solution for F as $F : n(x) = \sin\left(\frac{n\pi}{L}x\right)$

and for each of these a corresponding $G_n(t) = A_n \cos(\lambda_n t) + B_n \sin(\lambda_n t)$, where $\lambda_n = \frac{cn\pi}{L}$ and $A_n, B_n \in \mathbb{R}$

How to determine the remaining coefficients A_n, B_n?

Given our solutions for $n = 1, 2, ..$ as

$$u_n(x,t) = \big(A_n \cos(\lambda_n t) + B_n \sin(\lambda_n t)\big) \sin\big(\frac{n\pi}{L} x\big)$$

First, the wave equation $\frac{\partial^2}{\partial t^2} u = c^2 \frac{\partial^2}{\partial x^2} u$ is homogeneous!
Using the superposition principle the general solution reads

$$u(x,t) = \sum_{n=1}^{\infty} \big(A_n \cos(\frac{cn\pi}{L} t) + B_n \sin(\frac{cn\pi}{L} t)\big) \sin\big(\frac{n\pi}{L} x\big)$$

But even more: We have the initial conditions!

- $u(x,0) = f(x)$ for some given function $f(x)$ (initial position)
- $\frac{\partial}{\partial t} u(x,0) = g(x)$ for some given function $g(x)$ (initial velocity)

Summary: The Wave Eq. & Separation of Variables

For the one-dimensional wave equation

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} u = c^2 \dfrac{\partial^2}{\partial x^2} u & x \in [0, L] \\ u(x,0) = u(L,t) = 0 & \text{(boundary conditions)} \\ u(x,0) = f(x) & \text{(initial condition)} \\ \dfrac{\partial}{\partial t} u(x,0) = g(x) & \text{(initial condition)} \end{cases}$$

we obtained the solution by separation of variables as

$$u(x,t) = \sum_{n=1}^{\infty} \Big(A_n \cos(\frac{cn\pi}{L} t) + B_n \sin(\frac{cn\pi}{L} t) \Big) \sin(\frac{n\pi}{L} x)$$

with

$$A_n = \frac{2}{L} \int_0^L f(x) \sin(\frac{n\pi}{L} x) \mathrm{d}x$$

$$B_n = \frac{2}{cn\pi} \int_0^L g(x) \sin(\frac{n\pi}{L} x) \mathrm{d}x$$

Example.

We “lift” the center of the string to a height H and keep the velocity at 0 in the beginning We get

$$u(x,0) = f(x) = H\left(1 - \left|\frac{2x}{L} - 1\right|\right) = \begin{cases} \frac{2H}{L}x & \text{if } x \in [0, \frac{L}{2}) \\ \frac{2H}{L}(L-x) & \text{if } x \in [\frac{L}{2}, L] \end{cases}$$

$$\frac{\partial}{\partial t}u(x,0) = g(x) = 0, \qquad x \in [0, L]$$

What does the solution look like?