07 PDE

23/02/23

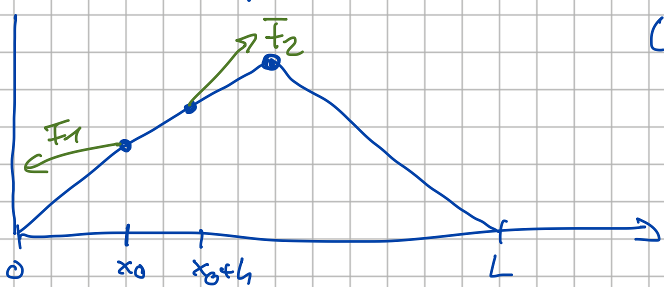
- Classification

- $$\frac{\partial u}{\partial y} - 3\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial t}\right)^2 = 0$$
  - nonlinear
  - homogeneous
  - order 2

- $$\frac{\partial^2 u}{\partial x^2} + \sin(u) - 1 = 0$$
  - nonlinear
  - nonhomogeneous
  - order 2

7-7, Derivation of the wave equation

(Initial szenario)



Someone is holding the string (still) in this position and at time $t_0 = 0$ the string is released $\rightsquigarrow$ How does the string vibrate?

We have forces acting at every point

Due to assumption 3 (only vertical movement) the horizontal components cancel out

$$F_1 = \begin{pmatrix} -T \\ -T_1 \end{pmatrix}, \quad F_2 = \begin{pmatrix} T \\ T_2 \end{pmatrix}, \quad \text{i.e. } (F_1)_x = -T = -(F_2)_x$$

By Newtons second law: any force F is mass times acceleration

$$F = m \cdot a, \qquad F, a \in \mathbb{R}^2, \quad m \in \mathbb{R}_+$$

So our vertical movement

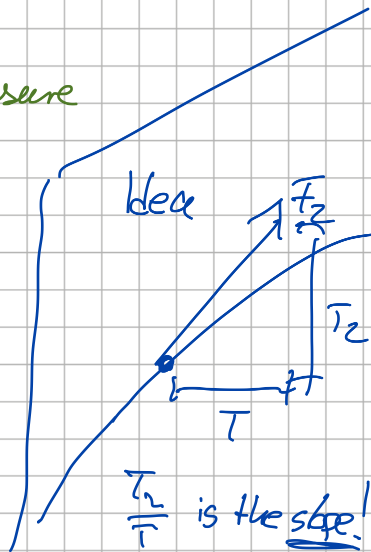
$$\underbrace{h}_{\text{length}} \cdot \underbrace{\rho}_{\text{mass per unit length}} \cdot \underbrace{\frac{\partial^2}{\partial t^2} u(x_0, t)}_{\text{acceleration}} = \underbrace{T_2 - T_1}_{\text{force we measure}}$$

mass

Divide by h : $$\rho \cdot \frac{\partial^2}{\partial t^2} u(x_0, t) = \frac{T_2 - T_1}{h}$$

Divide by the horizontal (canceling) force T

$$\frac{\rho}{T} \frac{\partial^2}{\partial t^2} u(x_0, t) = \frac{\frac{T_2}{T} - \frac{T_1}{T}}{h}$$



So $\frac{T_2}{T}$ is the slope of $u(x_0+h, t)$

$\frac{T_1}{T}$ slope at $u(x_0, t)$

But this is $\frac{\partial}{\partial x} u(x_0, t)$

$$\Rightarrow \quad \frac{\rho}{T} \frac{\partial^2}{\partial t^2} u(x_0, t) = \frac{\frac{\partial}{\partial x} u(x_0+h, t) - \frac{\partial}{\partial x} u(x_0, t)}{h}$$

This is a forward difference

Take $\lim_{h \to 0}$ of this equation

$$\Rightarrow \quad \underbrace{\frac{\rho}{T}}_{\frac{1}{c^2}} \frac{\partial^2}{\partial t^2} u(x_0, t) = \frac{\partial^2}{\partial x^2} u(x_0, t)$$

we introduce $c^2 = \frac{T}{\rho}$ (or $c = \sqrt{\frac{T}{\rho}}$

$$\Rightarrow \quad \frac{\partial^2}{\partial t^2} u(x, t) = c^2 \frac{\partial^2}{\partial x^2} u(x, t)$$

This is the wave equation.

Classification
- linear
- homogeneous
- order 2

7-9. For the wave equation we obtain

$$\frac{\partial^2}{\partial t} u = \frac{\partial^2}{\partial t^2} F(x)G(t) = F(x)\underline{G''(t)}$$

and

$$c^2 \frac{\partial^2}{\partial x} u = c^2 \frac{\partial^2}{\partial x^2} F(x)G(t) = c^2 F''(x)G(t)$$

In our PDE we get

$$\frac{\partial^2}{\partial t^2} u = \boxed{F(x)G''(t) = c^2 F''(x)G(t)} = c^2 \frac{\partial^2}{\partial x^2} u$$

Let's rearrange this

$$\underbrace{\frac{G''(t)}{c^2 G(t)}}_{\substack{\text{depends on } t\\ \text{constant in } x}} = \underbrace{\frac{F''(x)}{F(x)}}_{\substack{\text{depends on } x\\ \text{constant in } t}} = k$$

where $k \in \mathbb{R}$ is some constant

This means

$\boxed{1}$ $\frac{F''(x)}{F(x)} = k \iff F''(x) = kF(x) \iff \boxed{F''(x) - kF(x) = 0}$

$\boxed{2}$ $\frac{G''(t)}{c^2 G(t)} = k \iff \boxed{G''(t) - c^2 k G(t) = 0}$

These are 2 ODEs (ordinary differential equations)

We solve these in the following

7-10. The case $k = 0$.

We get (1) $F''(x) - kF(x) = F''(x) = 0$

$$\Rightarrow F'(x) = A$$

$$\Rightarrow F(x) = Ax + b$$

From the BC we know

$u(0,t) = F(0)G(t) = 0$ for all t

$u(L,t) = F(L)G(t) = 0$ for all t

First solution $G(t) = 0$ for all t

$u(x,t) = F(x)G(t) = 0$ for all x and t

a) this is boring

b) the IC $u(x,0) = f(x)$

if $f(x) \neq 0$ this is not a solution.

let's look for other solutions

Second solution $\underline{G(t)} \neq 0$ (at least for one t)

$\Rightarrow \quad F(0)\,\underline{G(t)} = 0 \qquad \Rightarrow F(0) = A \cdot 0 + B = 0$

$\Leftrightarrow \quad B = 0$

and $F(L)\,G(t) = 0 \quad \Rightarrow F(L) = A \cdot L + B = 0$

$\Leftrightarrow \quad A \cdot L = 0 \quad \Rightarrow A = 0$

$\Rightarrow F(x) = 0$ for $x \in [0,L]$

$\Rightarrow u(x,t) = F(x)G(t) = 0$

Summary: for $k=0$ only $u(x,t) = 0$ is a solution

this is not of interest.

7-11, The case $k>0$, let's say $k = \mu^2$, $\mu \in \mathbb{R} \setminus \{0\}$

(1) $F''(x) - k\,F(x) = 0 \Leftrightarrow \boxed{F''(x) = \mu^2 F(x)}$

(2) $G''(t) - c^2 k\,G(t) = 0 \Leftrightarrow G''(t) = (c\mu)^2 G(t)$

For F we take

$F(x) = e^{\mu x}$ or $F(x) = e^{-\mu x}$

$F'(x) = \mu e^{\mu x}$ $\quad$ $F'(x) = -\mu e^{-\mu x}$

$F''(x) = \mu^2 e^{\mu x}$ $\quad$ $F''(x) = \mu^2 e^{-\mu x}$

or a combination of both

$$F(x) = A e^{\mu x} + B e^{-\mu x}$$

Similarly (Check!)

$$G(t) = C e^{\mu c t} + D e^{-\mu c t}$$

(Again!) Check BC.

First solution: if $C = D = 0 \Rightarrow G(t) = 0$

(no solution of interest)

$G(t) \neq 0$ for some t.

$u(0,t) = F(0)\,G(t) = 0 \Rightarrow F(0) = 0$

$u(L,t) = 0 \Rightarrow F(L) = 0$

$F(0) = Ae^0 + Be^0 = A + B = 0 \qquad (\underline{A = -B})$

$F(L) = Ae^{\mu L} + Be^{-\mu L} = 0 \quad |$ divide by $e^{-\mu L}$ or mult. by $e^{\mu L}$

$\Rightarrow \quad Ae^{2\mu L} + B = 0$

$\Leftrightarrow \underbrace{(-1 + e^{2\mu L})}_{\neq 0} A = 0 \Rightarrow A = 0 \Rightarrow B = 0$

$\Rightarrow F(x) = 0 \qquad x \in [0, L]$

(no solution of interest)

7-12, (the last) case $k < 0$ or $k = -\lambda^2$ for $\lambda \in \mathbb{R} \setminus \{0\}$

① $F''(x) - kF(x) = 0 \Leftrightarrow \underline{F''(x)} = kF(x) = \underline{-\lambda^2 F(x)}$

(we look for functions whose 2nd derivative fulfills the eq.)

$F_3(x) = e^{-i\lambda x}$

$F_3'(x) = -i\lambda e^{i\lambda x}$

$F_3''(x) = -\lambda^2 e^{-i\lambda x}$

$F_1(x) = \sin(\lambda x)$

$F_1'(x) = \lambda\cos(\lambda x)$

$F_1''(x) = -\lambda^2 \sin(\lambda x) = -\lambda^2 F_1(x)$

as well as

$F_2(x) = \cos(\lambda x)$

$\Rightarrow F(x) = A\cos(\lambda x) + B\sin(\lambda x)$

(again) BC for $u(x,t) = F(x)G(t)$ ($G(t)$ is not the zero function)

① $u(0,t) = 0 \Rightarrow 0 = F(0) = A \cdot 1 + B \cdot 0 = \underline{A = 0}$

② $u(L,t) = 0 \Rightarrow 0 = F(L) = B\sin(\lambda L) = 0$

we look for zeros

can be nonzero

and $\sin(\lambda L)$ is zero if λL is a multiple of π

So we get $\lambda L = n\pi, \quad n = 1,2,3,\dots$

$\Leftrightarrow \lambda = \frac{n\pi}{L}$ (or our constant k has to be $k = -\left(\frac{n\pi}{L}\right)^2$)

then $B \cdot \sin\left(\frac{n\pi}{L}\right) = 0$ even for nonzero B!

and for every such λ we get one solution for F:

each $F_n(x) = B\sin\left(\frac{n\pi}{L}x\right)$ solves the first ODE

set $B=1$

So consider any $k = -\left(\frac{n\pi}{L}\right)^2$ in our second ODE

(2) $G''(t) = -\left(\frac{\pi n}{L}c\right)^2 G(t)$

Define $\lambda_n = \frac{\pi n}{L}c \quad \Rightarrow G''(t) = -\lambda_n^2 G(t)$

We get a solution for every λ_n as

$$G_n(t) = A_n\cos(\lambda_n t) + B_n \sin(\lambda_n t)$$

With that we finally have solutions

$$u_n(x,t) = F_n(x)\cdot G_n(t) = \left(A_n\cos(\lambda_n t) + B_n\sin(\lambda_n t)\right)\sin\left(\frac{n\pi}{L}x\right)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) \quad \text{is a solution}$$

Initial conditions, at $t=0$ $\cos(\lambda_n 0) = 1$ and $\sin(\lambda_n 0) = 0$

$$f(x) = u(x,0) = \sum_{n=1}^{\infty}\left(A_n\cdot 1 + \cancel{B_n\cdot 0}\right)\sin\left(\frac{n\pi}{L}x\right) \quad x\in \text{ on } [0,L)$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) \quad x\in[0,L]$$

use the odd extension of f $\quad f_o(x),\ x\in[-L,L]$

$$\Rightarrow A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx$$

For the B_n we use $\quad u_n(x,t) = \left(A_n \cos(\lambda_n t) + B_n \sin(\lambda_n t)\right) \cdot \sin\left(\frac{n\pi}{L}x\right)$

$$\frac{\partial}{\partial t} u(x,0) = \sum_{n=1}^{\infty} \left(-A_n \cdot 0 + B_n \frac{cn\pi}{L}\right) \sin\left(\frac{n\pi}{L}x\right)$$

$$= g(x) \qquad \text{(initial condition)}$$

If we call $\quad d_n = B_n \frac{cn\pi}{L}$

$$g(x) = \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi}{L}x\right)$$

use the odd extension $g_o(x) \Rightarrow d_n = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx$

$$\Rightarrow B_n = \frac{L}{cn\pi} d_n = \frac{2}{cn\pi}\int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx$$