6 - Fourier Transforms

16/02/23

For $f: \mathbb{R} \to \mathbb{C}$ compute the c_k of

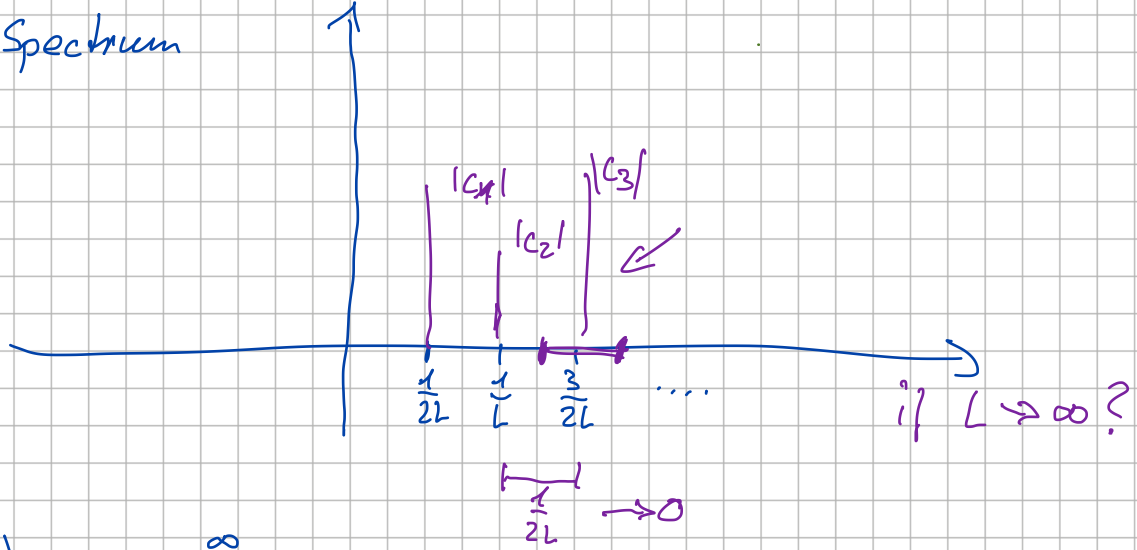
$$\left(f\big|_{[-L,L]}\right)(x) \sim \sum_{k=-\infty}^{\infty} \underbrace{\frac{1}{2L} \int_{-L}^{L} f(y) e^{-\frac{iky\pi}{L}} dy}_{c_k\left(f|_{[-L,L]}\right)} \; e^{\frac{ikx\pi}{L}}$$

f on $[-L,L]$

(ignore the rest)

We introduce $\Delta\omega = \frac{\pi}{L}$ and $\omega_k = k \cdot \Delta\omega = \frac{k\pi}{L}$

Remember Spectrum



$$\left(f\big|_{[-L,L]}\right)(x) \sim \sum_{k=-\infty}^{\infty} \Delta\omega \cdot \underbrace{\frac{1}{2\pi} \int_{-L}^{L} f(y)\, e^{i\omega_k y}\, dy}_{=:\sqrt{2\pi}\,\hat{f}_L(\omega_k)} \; e^{ix\omega_k}$$

and we get

$$\sim \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \Delta\omega\, \hat{f}_L(\omega_k)\, e^{i\omega_k x}$$

Riemann sum $L \to \infty$

For $L \to \infty$ we have $\Delta\omega \to 0$ $\omega_k \to \omega$ $\hat{f}_L \to \hat{f}$ as follows

$$f(x) \sim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, e^{-i\omega y}\, dy}_{=\hat{f}(\omega)} \; e^{i\omega x}\, dx$$

6-8, We want to compute $\hat{f}(\omega)$ of $f(x) = \chi_{[a,b]}(x)$

Start with

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi_{[a,b]}(x)\, e^{-i0x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b 1 \cdot 1 \, dx = \frac{b-a}{\sqrt{2\pi}}$$

Let $\omega \neq 0$ be given

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi_{[a,b]}(x)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b 1 \cdot e^{-i\omega x}\, dx =$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{-\omega i} e^{-i\omega x} \Big|_{x=a}^{b}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{-i\omega} \left(e^{-i\omega b} - e^{-i\omega a} \right) \cdot e^{i\omega\left(\frac{a+b}{2}\right)} \cdot e^{-i\omega\left(\frac{a+b}{2}\right)}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1 \cdot 2}{\omega} \left(\frac{e^{i\omega\left(\frac{b-a}{2}\right)} - e^{i\omega\left(\frac{b-a}{2}\right)}}{-i \cdot 2} \right) e^{i\omega\left(\frac{a+b}{2}\right)}$$

$$= \sqrt{\frac{2}{\pi}} \; \frac{\sin\left(\frac{\omega(b-a)}{2}\right)}{\omega} \cdot e^{i\omega\left(\frac{a+b}{2}\right)}$$

Special case $a = -1 \quad b = 1 \Rightarrow \frac{b-a}{2} = 1, \quad \frac{a+b}{2} = 0$

$$\Rightarrow f(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{else} \end{cases} \qquad \hat{f}(\omega) = \sqrt{\frac{2}{\pi}} \underbrace{\frac{\sin(\omega)}{\omega}}_{=:\,\mathrm{sinc}(\omega)}$$

Analogously

$f(x) = \mathrm{sinc}(x)$ yields $\hat{f}(\omega) = \begin{cases} \sqrt{\frac{\pi}{2}} & \text{if } |\omega| \leq 1 \\ 0 & \text{else} \end{cases}$

6-10

$$\int f' g = fg\Big| - \int f g'$$

$$\sqrt{2\pi}\,\mathcal{F}(f') = \lim_{L\to\infty} \int_{-L}^{L} f'(x) e^{-i\omega x}\, dx$$

$g(x) = e^{-i\omega x}$ $\qquad$ $g'(x) = -i\omega e^{-i\omega x}$

$$= \lim_{L\to\infty} \underbrace{f(x)\cdot e^{-i\omega x}\Big|_{x=-L}^{L}} - \int_{-L}^{L} f(x)(-i\omega) e^{-i\omega x}\, dx$$

$f(x) \to 0$ when $x \to \pm\infty$

so this term is zero

$$= +i\omega \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx = i\omega \sqrt{2\pi}\, \mathcal{F}(f)$$

If f and f' are "nice enough", do this twice

$$\Rightarrow \mathcal{F}(f'') = -\omega^2 \mathcal{F}(f)$$