TMA4125 Matematikk 4N

Fourier Series.

Ronny Bergmann

Department of Mathematical Sciences, NTNU.

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Motivation for periodic functions

We would like to investigate periodic phenomena, for example

- alternating current
- heart beat
- water waves
- audio & audio processing

or in general phenomena that repeat.

This might also happen for complex-valued functions $f\colon \mathbb{R} \to \mathbb{C}$.

History – Jean-Baptiste Fourier & trigonometric series

Jean-Baptiste Joseph Fourier
(1768 – 1830)

- First trigonometric series:
  Euler (1750), D. Bernoulli (1753)
- Fourier (1807):
  Propagation of Heat in Solid Bodies
- He postulated:

“Every periodic function
can be written as
a superposition
of trigonometric functions”

Periodic functions

Definition. A function $f\colon \mathbb{R} \to \mathbb{C}$ is called periodic if for some $T > 0$ it holds

$$f(x) = f(x+T) \qquad \text{for all } x \in \mathbb{R}.$$

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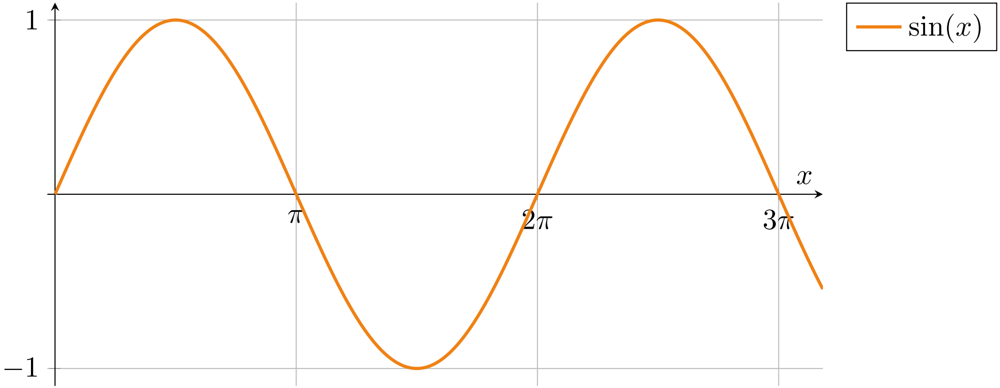
We will usually consider functions with fundamental period $T = 2\pi$.

Periodic functions – Examples.

Examples. What fundamental periods do the following functions have?



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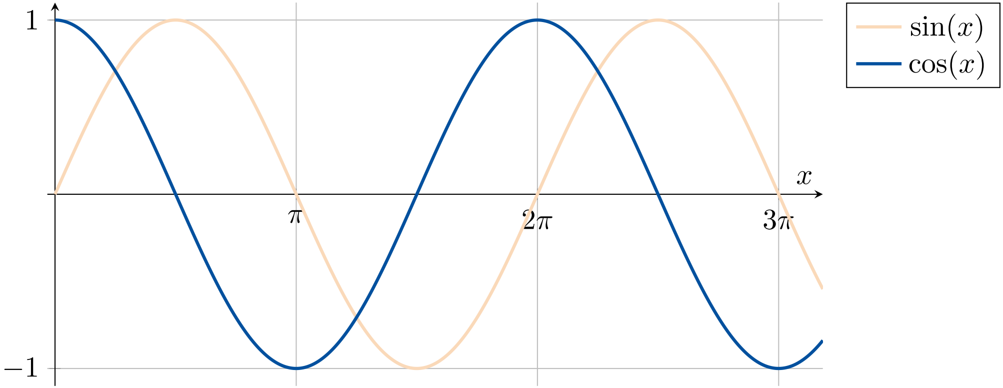

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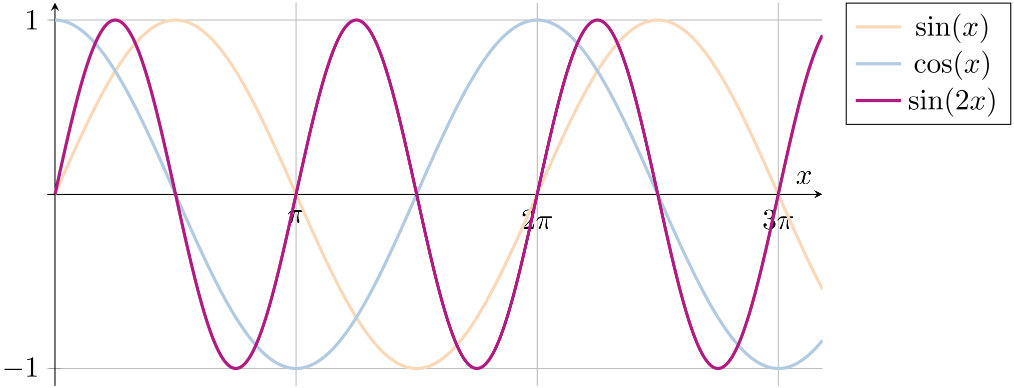

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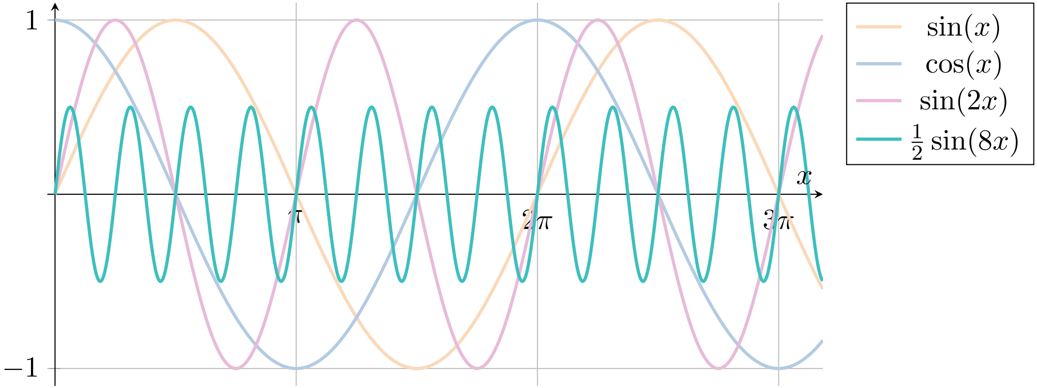

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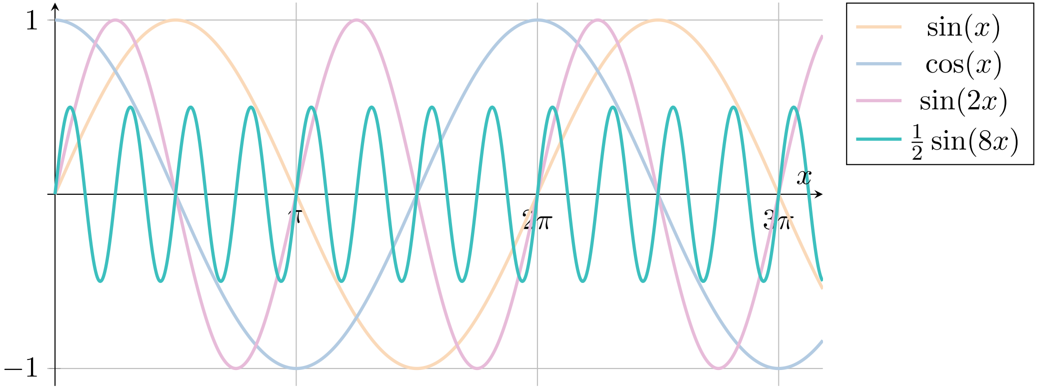
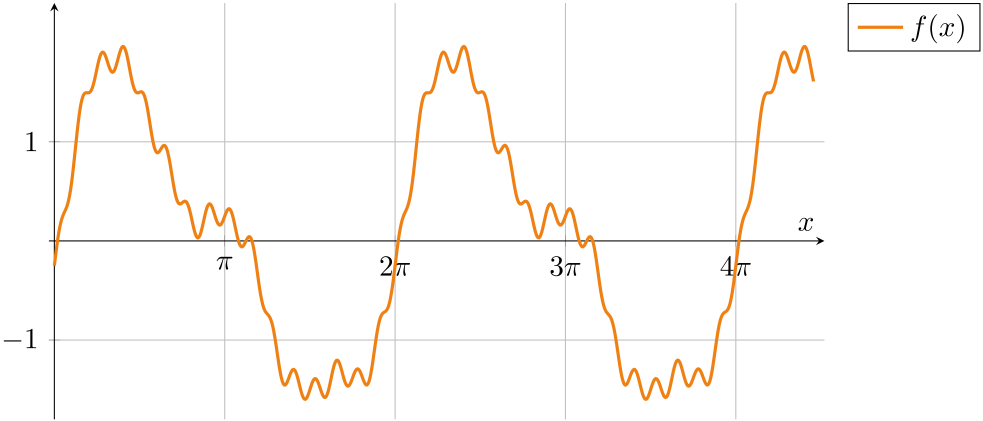


What about $f(x) = \sin(nx)$ for some $n \in \mathbb{N}\backslash\{0\}$?
fundamental period T?
Frequency in $T = 2\pi$?

Decomposing functions

- The following function f is $T = 2\pi$ periodic and looks a little chaotic.



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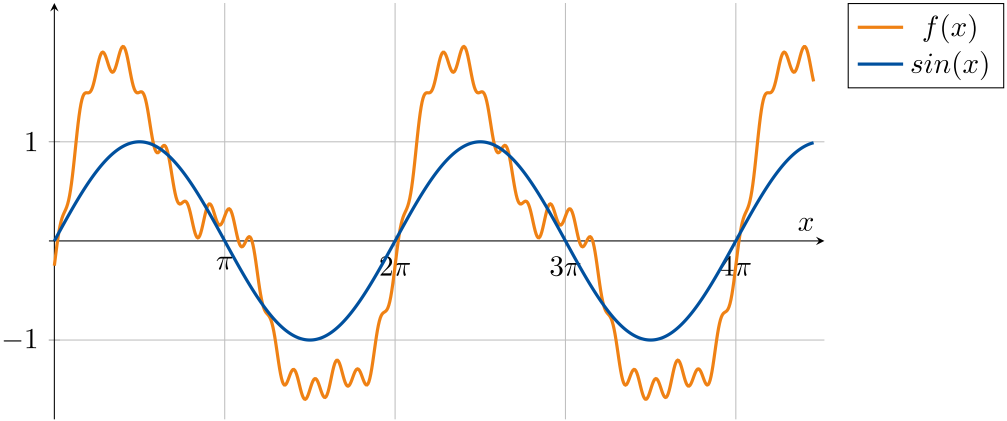

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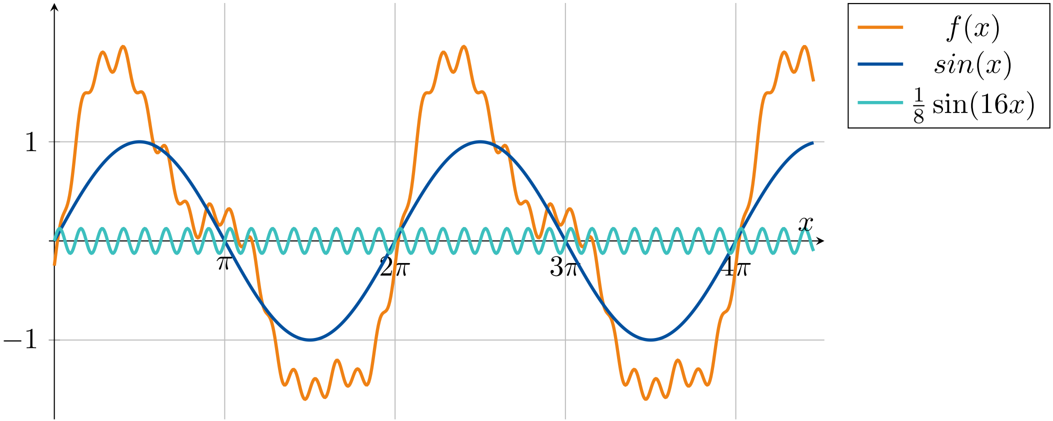

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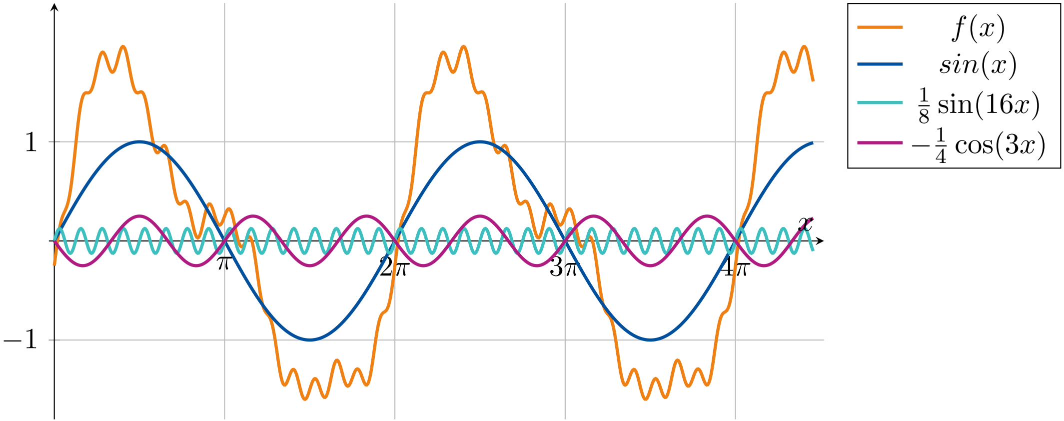

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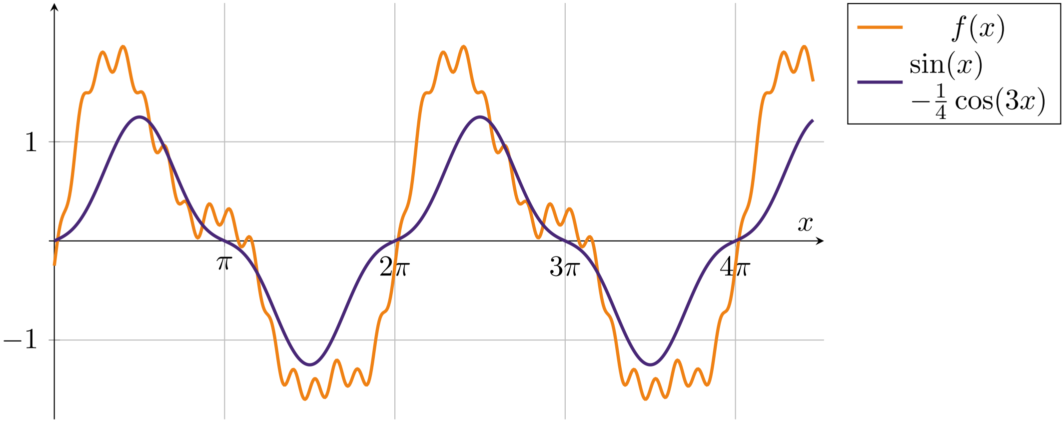

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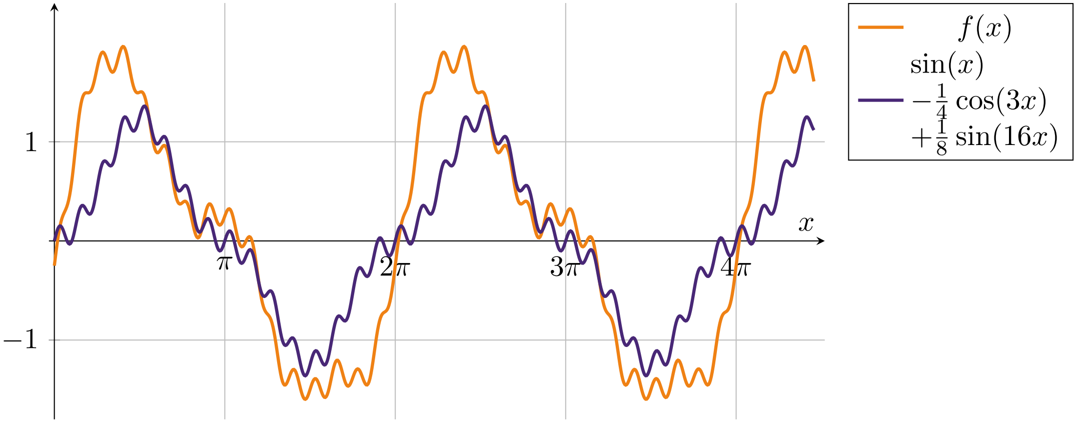

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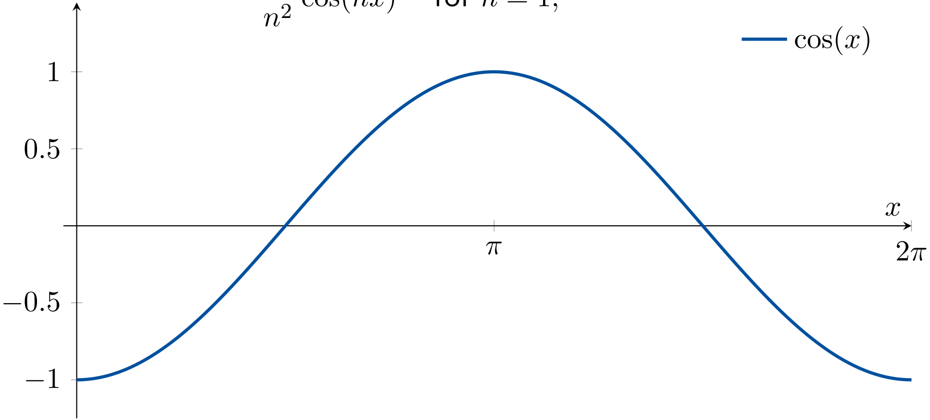

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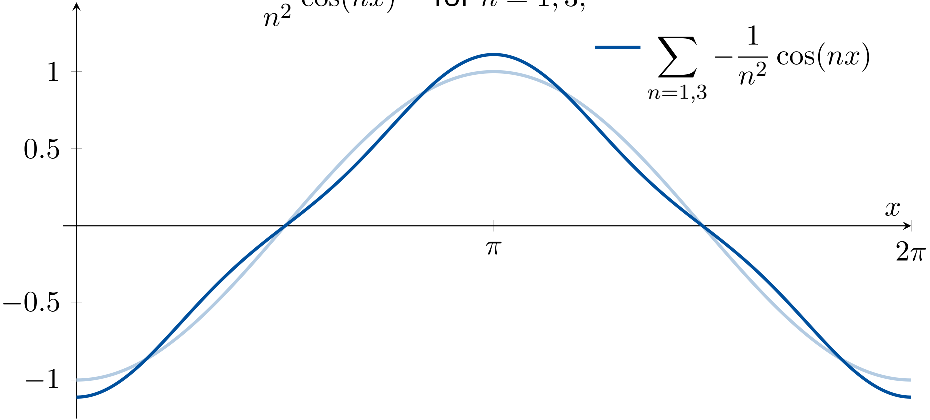

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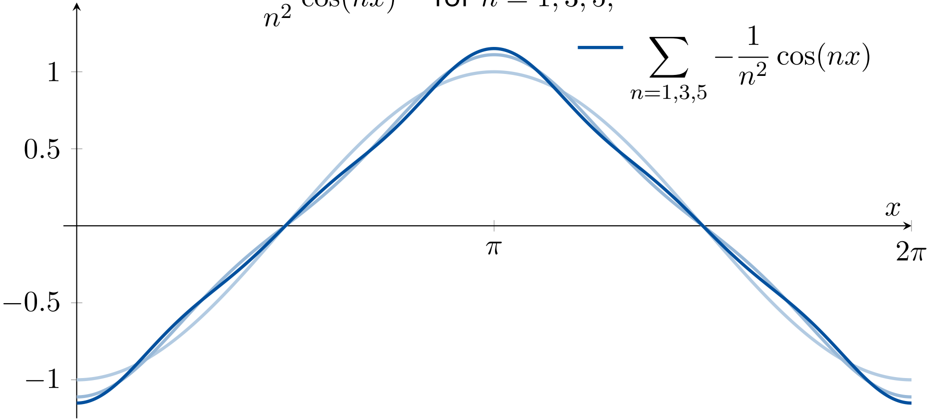

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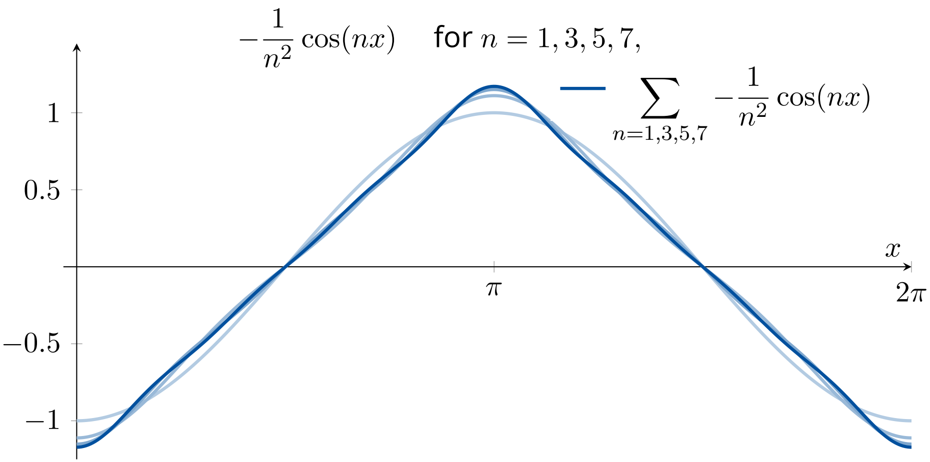

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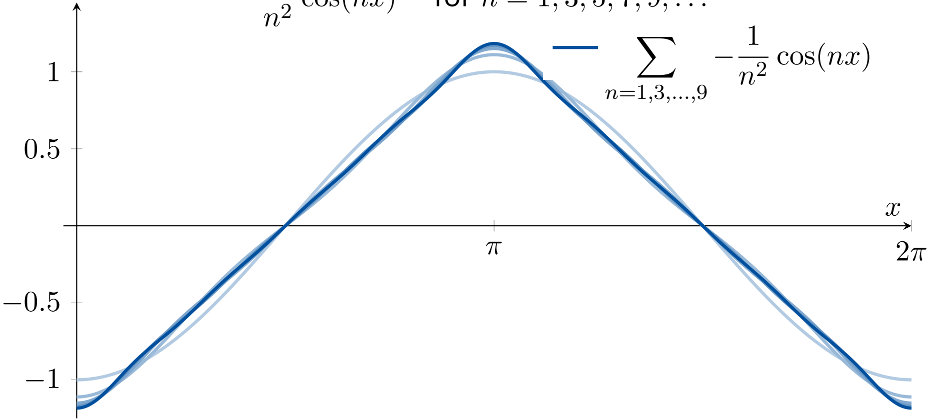

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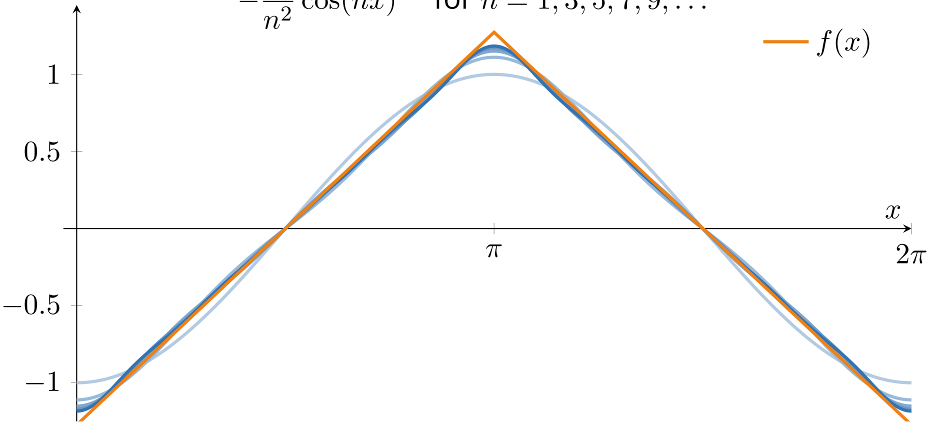

Fourier Series. A general construction scheme

Idea. use $\sin(nx)$ and $\cos(nx)$ for $n \in \mathbb{N}$ and $\cos(0x) = 1$

"to construct functions".

For coefficients a_0, a_n, b_n, $n = 1, 2, \ldots$ (real or complex),
we call the function

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Question. What happens in the limit $\lim\limits_{N\to\infty} f_N(x)$?

Reminder. Inner product

Let V be a complex vector space. Then a mapping $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ is called an inner product if the following properties hold

1. Linearity in the first argument: for all $f, g, h \in V$, $\alpha, \beta \in \mathbb{C}$

$$\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$$

2. Conjugate symmetry: for $f, g \in V$ we have

$$\langle f, g \rangle = \overline{\langle g, f \rangle}$$

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Example. Let V be the space of complex continuous 2π periodic functions. $f\colon [0, 2\pi) \to \mathbb{C}$. Then

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\,\mathrm{d}x$$

is a complex inner product on V.

Reminder. Euler formula

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- and we also have e. g.
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- But keep in mind the representation as $\mathrm{e}^{\mathrm{i}x}$ is a complex function.

The space of trigonometric polynomials up to degree N.

We denote by

$$V_N := \left\{ g \;\middle|\; g = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx), \quad a_0, a_1, \ldots, a_N, b_1, \ldots, b_N \in \mathbb{C} \right\}$$

the (real) vector space of periodic functions of degree at most N (or functions that can be "composed" by sine and cosine up to frequency N)

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$$\tilde{V}_N := \left\{ g \;\middle|\; g = \sum_{k=-N}^{N} c_k \mathrm{e}^{\mathrm{i}kx}, \quad c_{-N}, c_{-N+1}, \dots, c_N \in \mathbb{C} \right\}$$

the complex vector space of trigonometric polynomials of degree at most N.

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the complex vector space of trigonometric polynomials of degree at most N.
...and see directly that $V_N = \tilde{V}_N$.

A norm on a complex vector space

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Question. When/how can we “describe” a 2π-periodic function f “best possible” by a function $g \in V_N$, i. e. such that

$$\|g - f\| \quad \text{is as small as possible (among all } g \in V_N\text{)?}$$

Orthogonal and orthonormal systems

A sequence/family $\{\varphi_n\}_n$ of functions is called

- orthogonal if

$$\langle \varphi_k, \varphi_n \rangle = \begin{cases} 0 & \text{if } k \neq n \\ d_n \neq 0 & \text{if } k = n, \end{cases}$$

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$\Rightarrow$ We can create an orthonormal system from an orthogonal one by rescaling $\tilde{\varphi}_k = \frac{1}{\sqrt{d_k}} \varphi_k$

The trigonometric system is orthonormal

Remember. for (complex) 2π periodic functions: $\langle f, g\rangle = \int_{\pi}^{\pi} f(x)\overline{g(x)}\,\mathrm{d}x$.

Example 1. the sequence $\varphi_k = \mathrm{e}^{\mathrm{i}kx}$, $k \in \mathbb{Z}$, is an orthogonal system.

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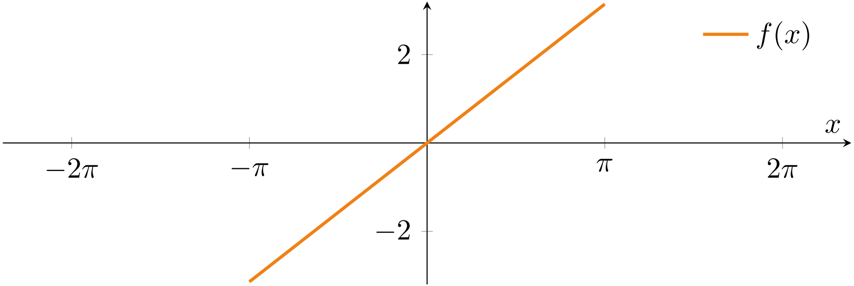
Example 2. We consider the family $\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots\}$
or shorter $\{\sin(kx)\}_{k=1}^{\infty} \cup \{\cos(kx)\}_{k=0}^{\infty}$.
These functions also form a orthogonal system

How to “best describe” f using V_N?

Let f be a 2π-periodic function, which function is “closest” to f from within V_N, i. e. which function $g \in V_N$ minimises $\|g - f\|$?

Example. We can define a 2π periodic function f by just defining it on one interval of length 2π and continue the rest:

$$f(x) = x, \quad \text{for } x \in [-\pi, \pi)$$



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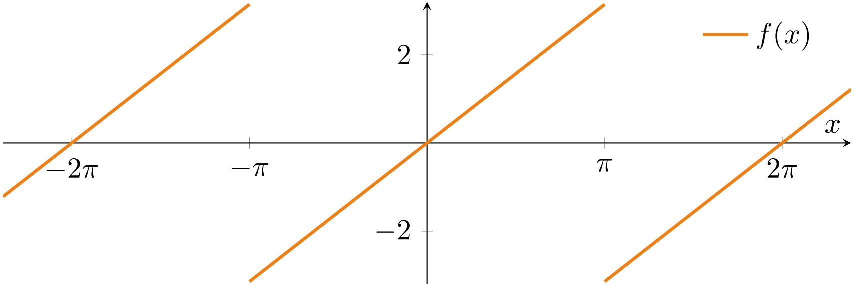

How to “best describe” f using V_N?

Let f be a 2π-periodic function, which function is “closest” to f from within V_N, i. e. which function $g \in V_N$ minimises $\|g - f\|$?

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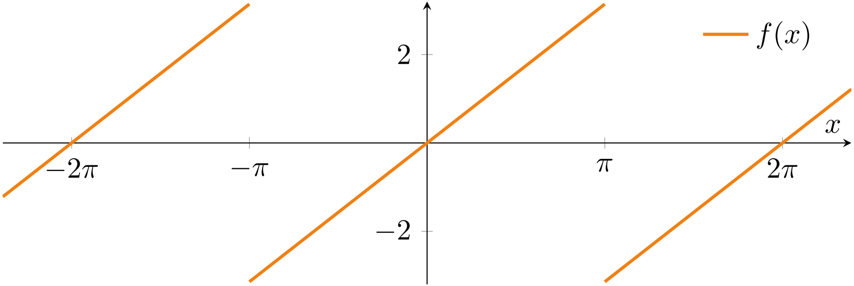

...and how to best compute g (i. e. coefficients a_n, b_n or the c_ks)?

The simpler case: $g = f \in V_N$ – Compute the coefficients.

Assume. In the following let V be a complex vector space with inner product $\langle \cdot, \cdot \rangle$ and an orthogonal system $\{\varphi_k\}_k \subset V$.

NTNU

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Even shorter. If $\{\varphi_k\}_k$ is an orthonormal system, then $c_k = \langle f, \varphi_k \rangle$.

(General) Fourier coefficients

Definition. For an orthogonal system $\{\varphi_k\}_k \subset V$ and a given vector/function $f \in V$ the coefficients

$$c_k = \frac{\langle f, \varphi_k \rangle}{\|\varphi_k\|^2}.$$

are called (general) **Fourier coefficients**.

Sometimes we write $\hat{f}(k) = c_k$

Orthogonal Projection & best approximation

For $f \in V$ we define the **projection** $\Pi_N f \in V_N$ by

$$\Pi_N f = \sum_{k=1}^{N} c_k \varphi_k \quad \text{with} \quad c_k = \frac{\langle f, \varphi_k \rangle}{\|\varphi_k\|^2}.$$

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Theorem. Let $f \in V$ be given. Then $\Pi_N f \in V_N$ satisfies

$$\|f - \Pi_N f\| = \min_{g \in V_N} \|f - g\|.$$

General Fourier Series

Definition. (General Fourier series for an orthogonal system)
Let $\{\varphi_k\}_{k=1}^{\infty}$ be an orthogonal system. Then the formal expression

$$\sum_{k=1}^{\infty} \hat{f}(k)\varphi_k, \qquad \hat{f}(k) = \frac{\langle f, \varphi_k \rangle}{\|\varphi_k\|^2}$$

is called a **general Fourier series**.

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For the specific case for periodic functions we still need

$$V := \left\{ f\colon [-\pi,\pi) \to \mathbb{C} \;\middle|\; \int_{-\pi}^{\pi} |f(x)|^2 \,\mathrm{d}x < \infty \right\},$$

where we assume that the integral exists.
Functions $f \in V$ are said to be square-integrable.
V is indeed a vector space.

Complex trigonometric series / Fourier series

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Let f be a 2π-periodic function and consider the orthogonal system

$$\left\{\mathrm{e}^{\mathrm{i}kx}\right\}_{k\in\mathbb{Z}}$$

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We also write $c_k(f)$ or $\hat{f}(k)$ for the c_k.

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We write this association also as

$$f \sim \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathrm{i}kx}$$

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Let f be a 2π-periodic function and consider the orthogonal system

$$\{1\} \cup \{\cos(nx)\}_{n\in\mathbb{N}} \cup \{\sin(nx)\}_{n\in\mathbb{N}}$$

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$$f \sim a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

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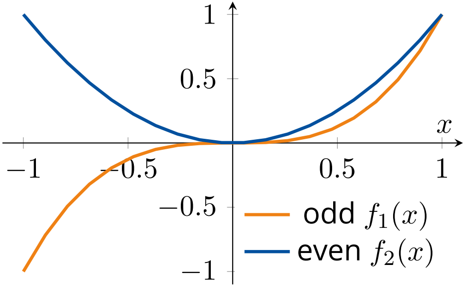
Note. a_0 and a_n,b_n, $n \in \mathbb{N}$, are all real values $\Leftrightarrow$ f is real valued.

Reminder. Even and odd functions

A function is called even if $f(x) = f(-x)$ holds for all x.
A function is called odd if $f(x) = -f(-x)$ holds for all x.



we have the following

f	g	$f \cdot g$
even	even	even
odd	even	odd
even	od	odd
odd	odd	even

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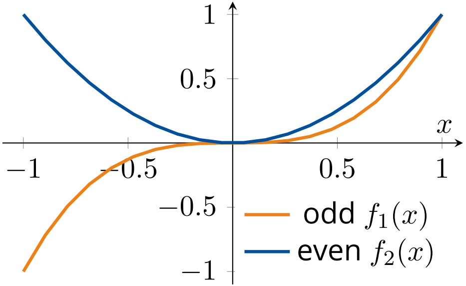

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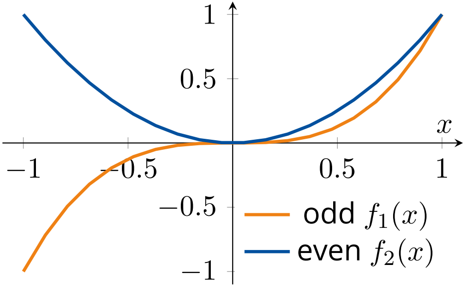

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Even better.

- if f is odd then $\int_{-L}^{L} f(x)\,\mathrm{d}x = 0$
- if f is even then $\int_{-L}^{L} f(x)\,\mathrm{d}x = 2\int_{0}^{L} f(x)\,\mathrm{d}x$

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We can use Eulers formula to relate the (complex) Fourier coefficients c_k and the (real) Fourier coefficients a_0, a_n, b_n:

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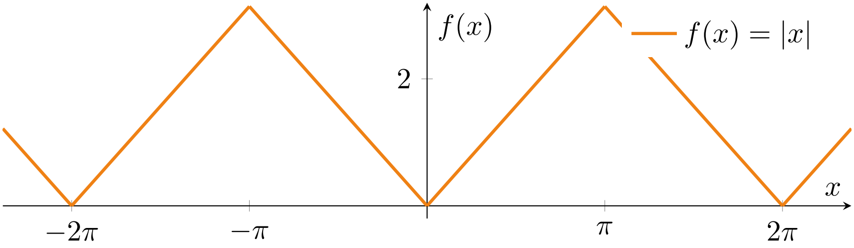
and even more: the projection onto $V_N = \tilde{V}_N$ yield the same partial sum

$$S_N(f) = \sum_{k=-N}^{N} c_k \mathrm{e}^{\mathrm{i}kx} = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx).$$

$\Rightarrow$ Choose the coefficients that "fit better".

Example 1. Fourier coefficients of $f(x) = |x|$, $x \in [-\pi, \pi)$.



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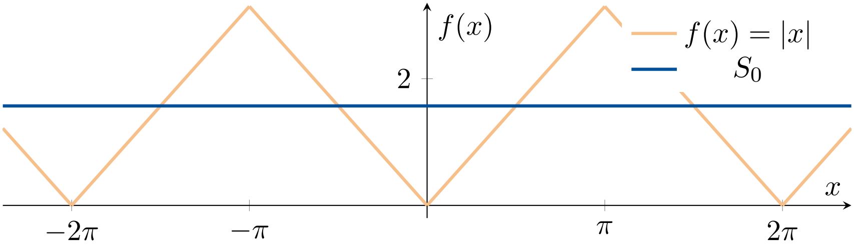

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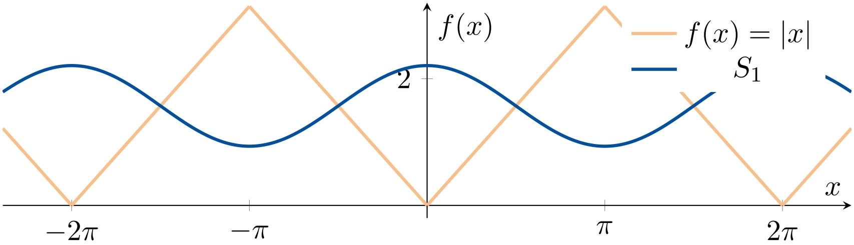

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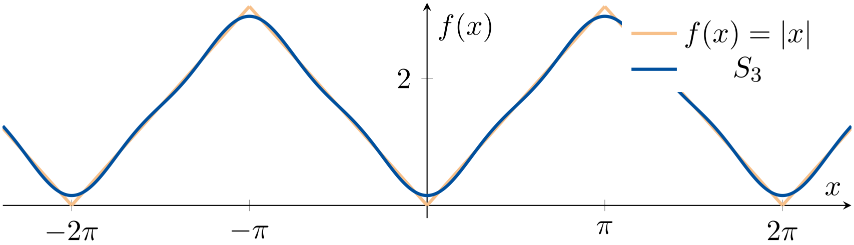

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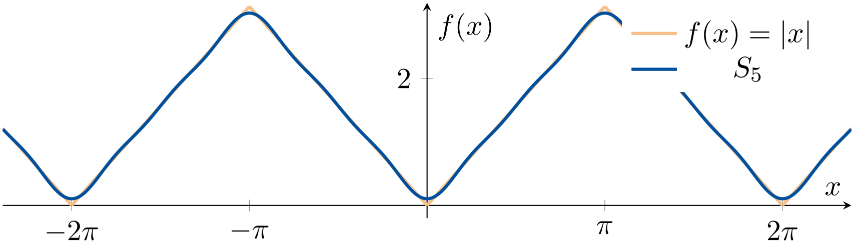

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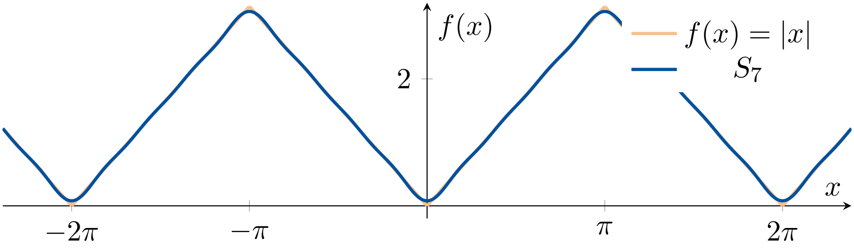

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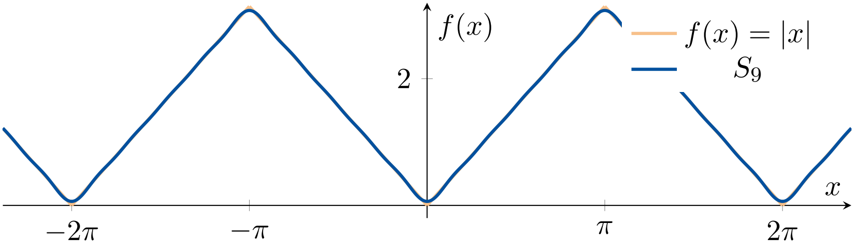

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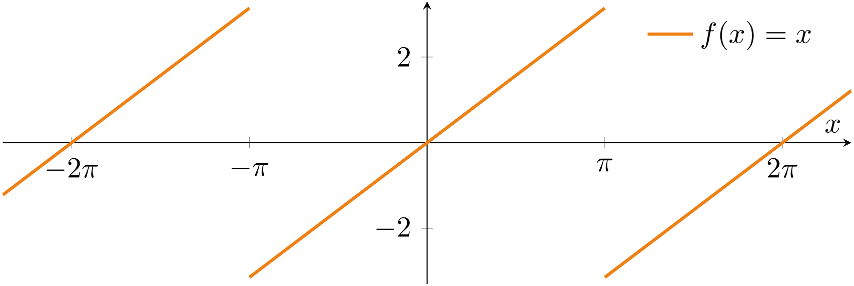

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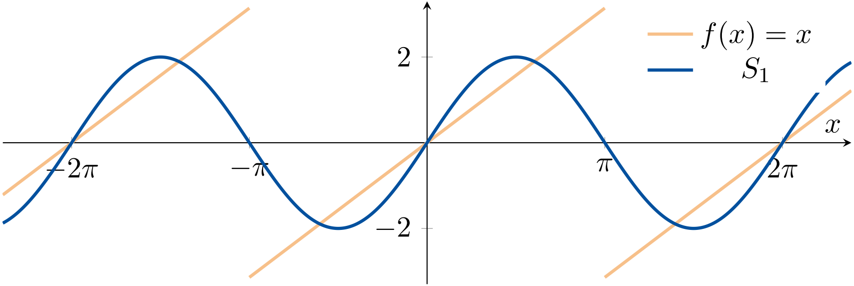

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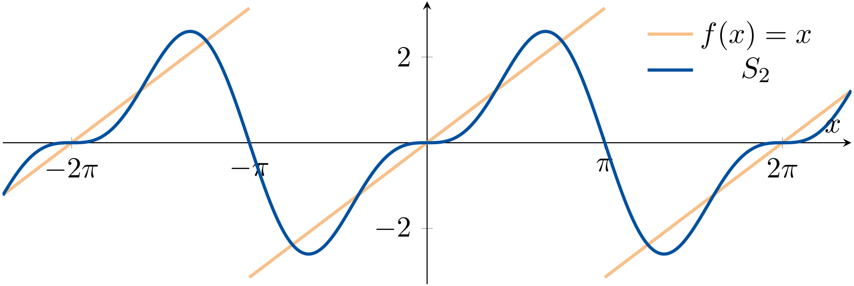

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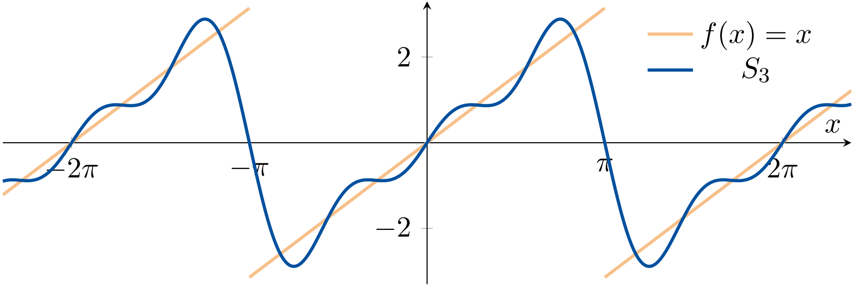

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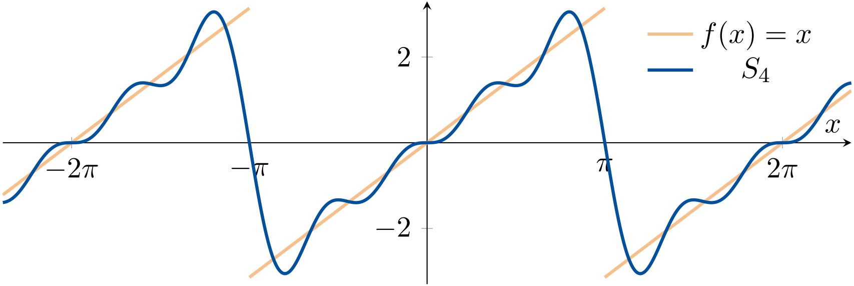

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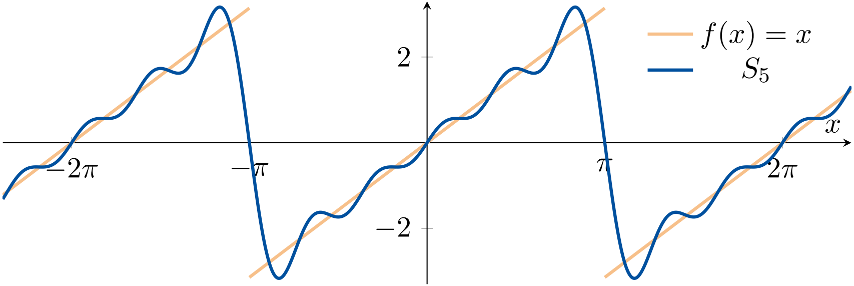

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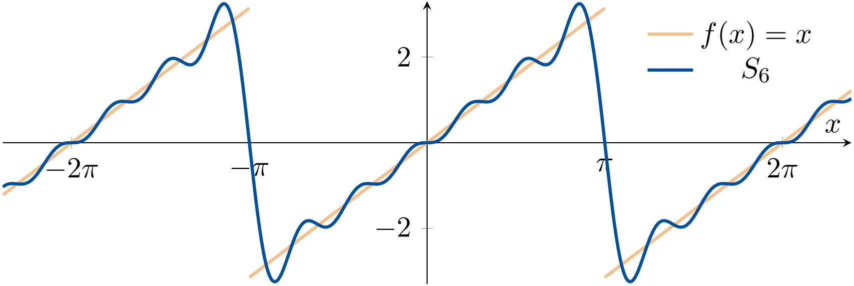

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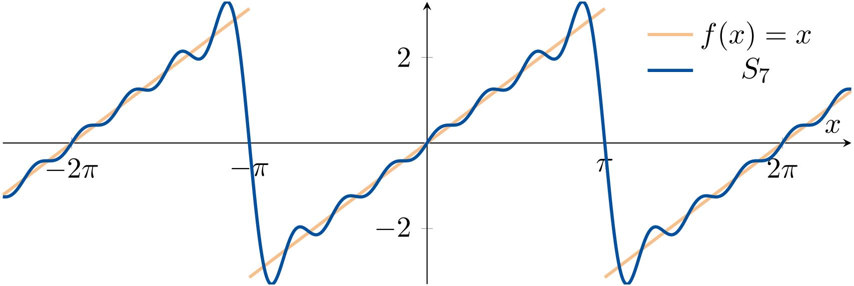

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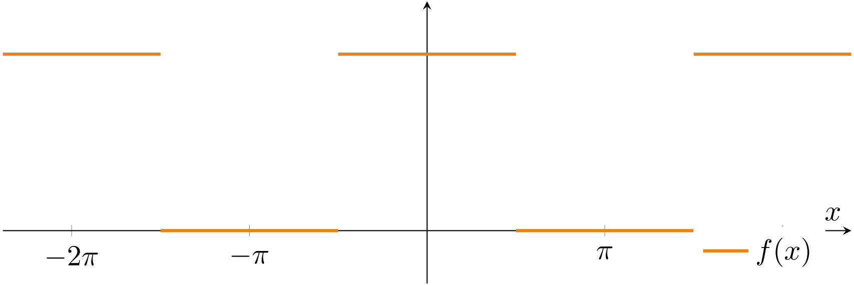
(Shifted) Heaviside function

Consider the function $f(x) = \begin{cases} 1 & \text{for } |x| \leq \frac{\pi}{2} \\ 0 & \text{else.} \end{cases}$



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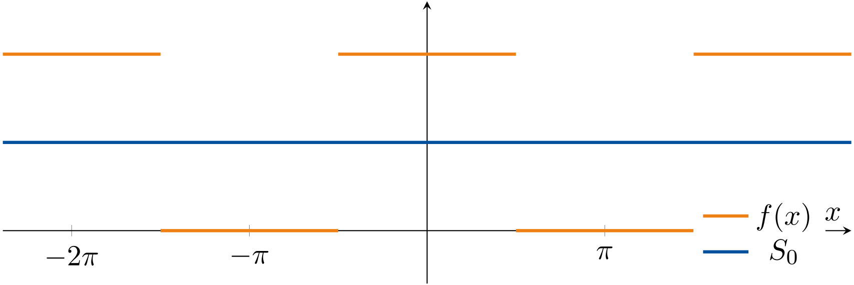

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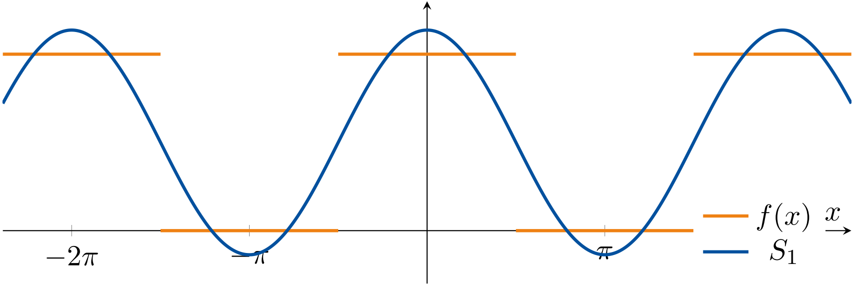

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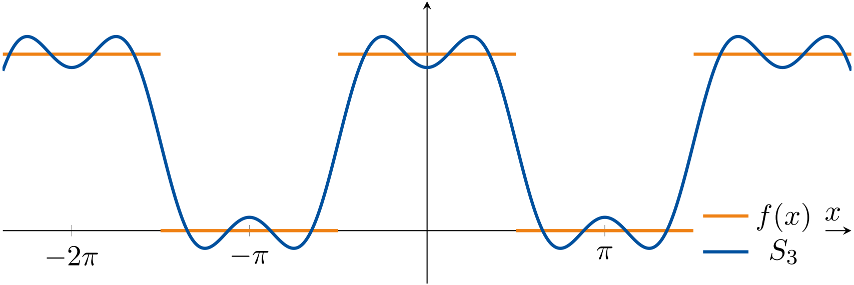

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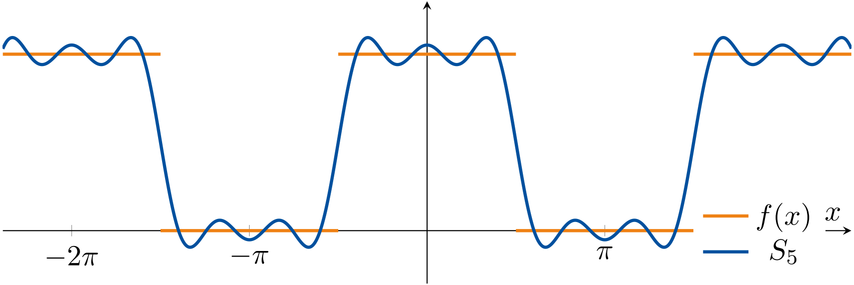

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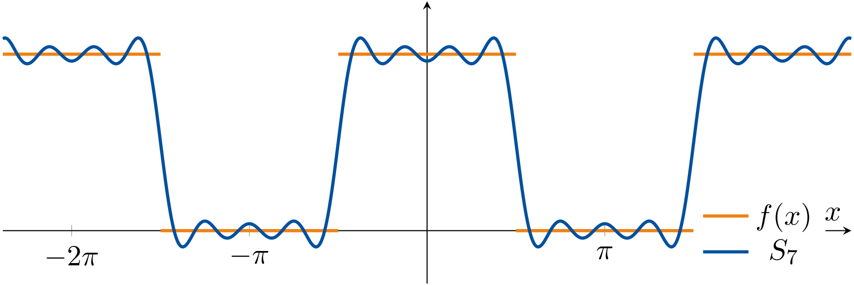

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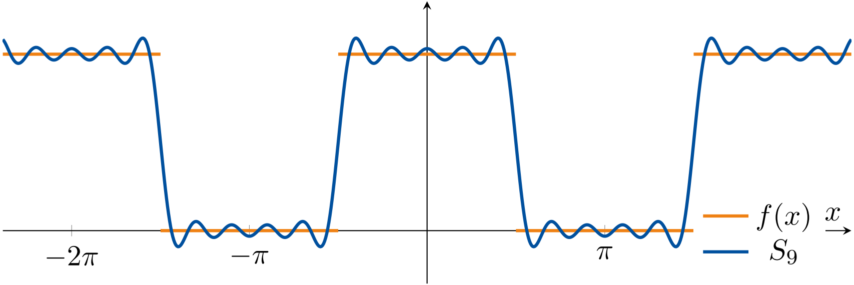

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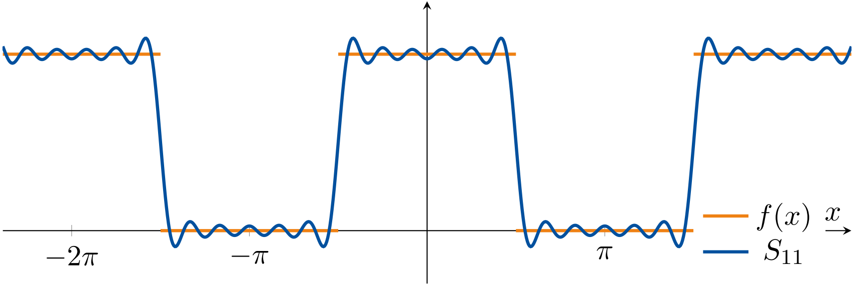

...but let's turn to Python to really see the effect.

Convergence – Motivation

We introduced the Fourier series as

$$f \sim \sum_{k=-\infty}^{\infty} \hat{f}(k)\mathrm{e}^{\mathrm{i}kx}$$

with

$$\hat{f}(k) = \langle f, \mathrm{e}^{\mathrm{i}kx} \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x$$

Question. When/For which x does

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\mathrm{e}^{\mathrm{i}kx} = \lim_{N\to\infty} S_N f(x)$$

hold?

Remember that $S_N f$ is the Fourier partial sum

$$S_N f(x) = \sum_{k=-N}^{N} \hat{f}(k)\mathrm{e}^{\mathrm{i}kx}$$

The Dirichlet kernel

The Nth Dirichlet kernel is defined by

$$D_N(x) = \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}kx}$$

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$$\int_{-\pi}^{\pi} D_N(x)\,\mathrm{d}x = 2\pi.$$

We can define the normalized Dirichlet Kernel

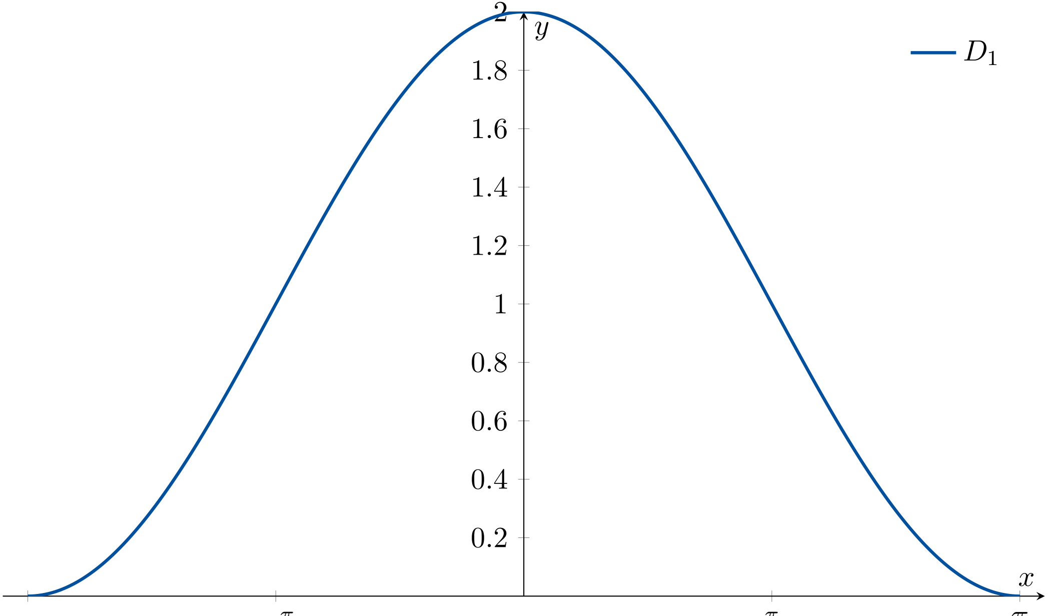
$$\tilde{D}_N(x) = \frac{1}{2\pi} D_N(x) \quad \text{where we get} \quad \int_{-\pi}^{\pi} \tilde{D}_N(x)\,\mathrm{d}x = 1$$

Rewriting the Dirichlet kernel

Using a shift in the sum index and the geometric sum with $q = \mathrm{e}^{\mathrm{i}x}$ we can rewrite

$$D_N(x) = \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}kx} = \frac{\sin\bigl((N+\frac{1}{2})x\bigr)}{\sin\bigl(\frac{x}{2}\bigr)}$$

Plotting the Dirichlet Kernel



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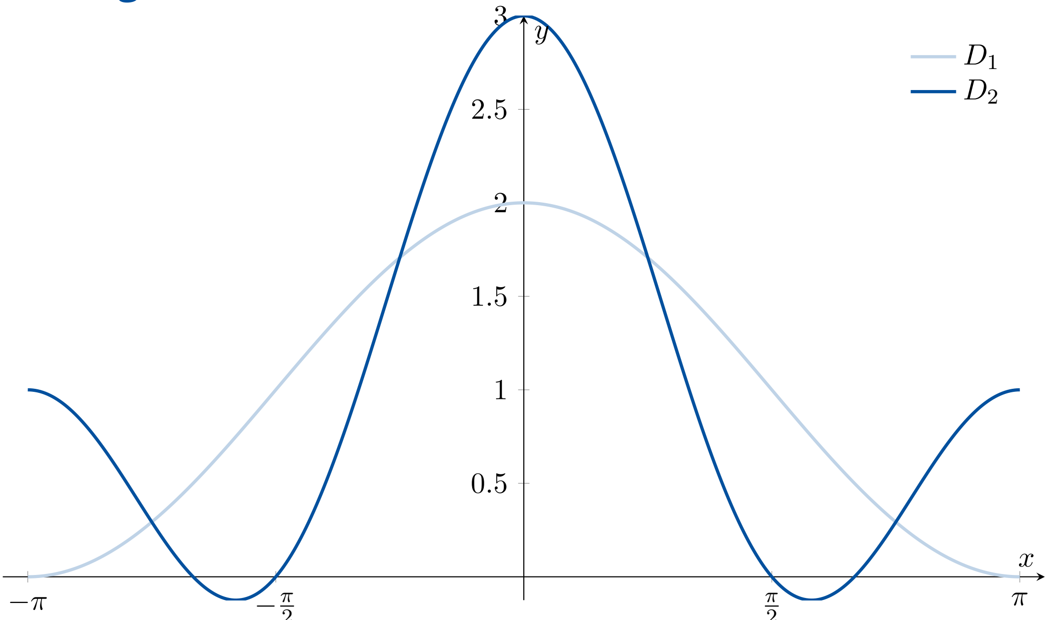

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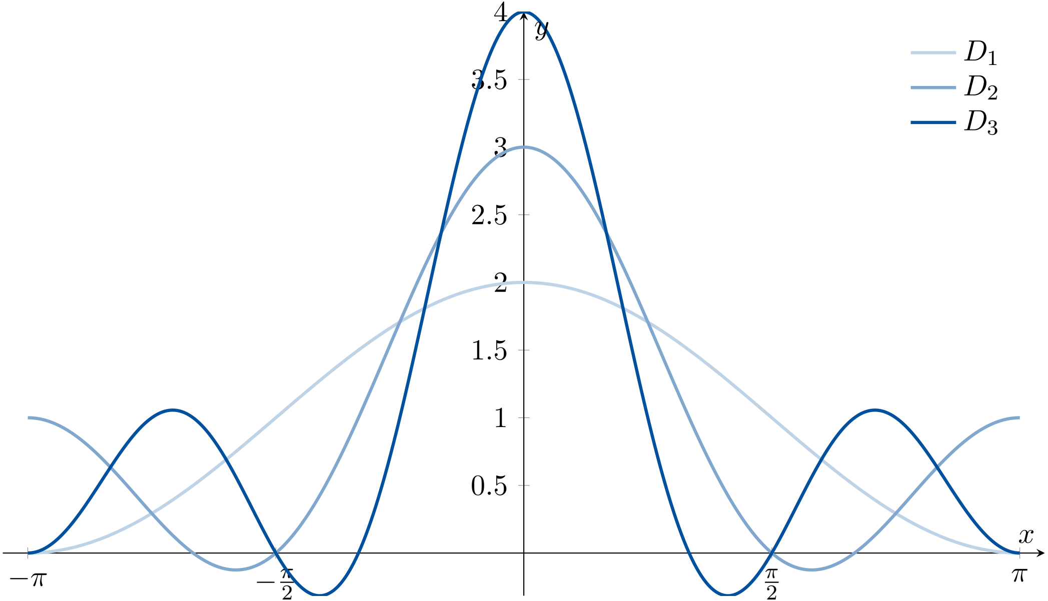

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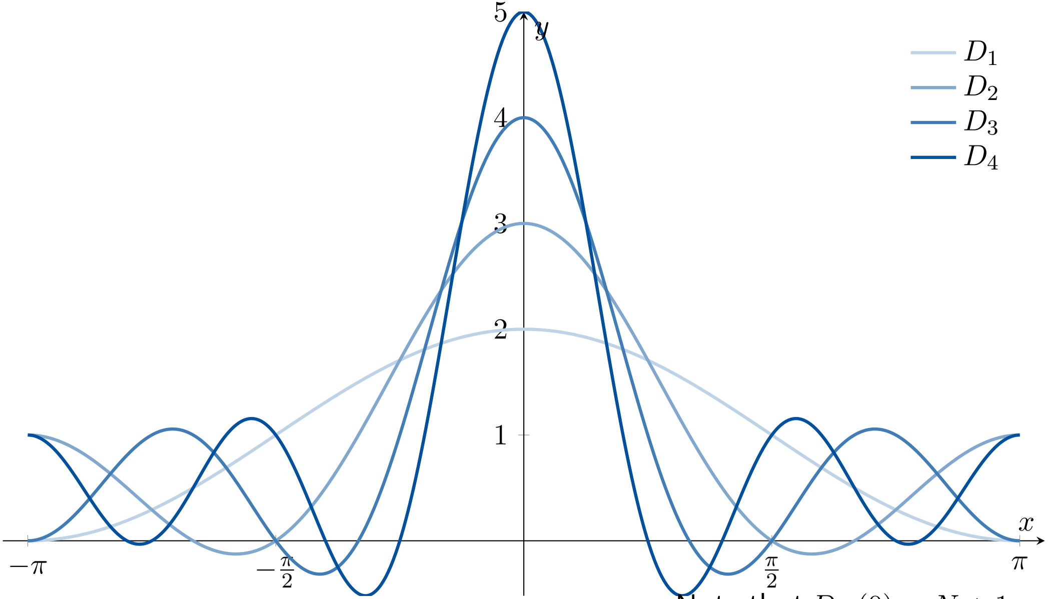

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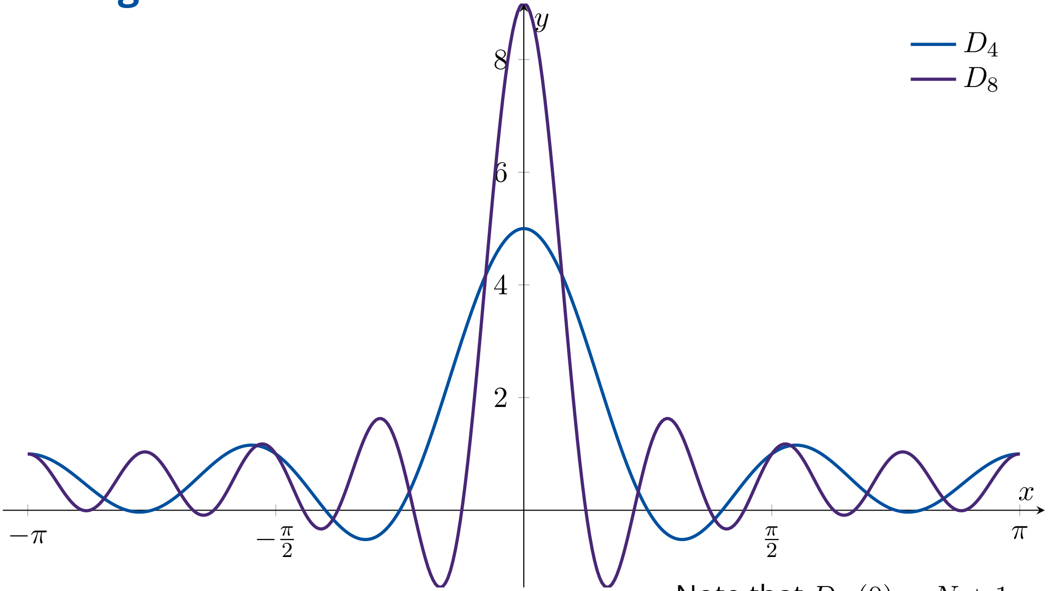

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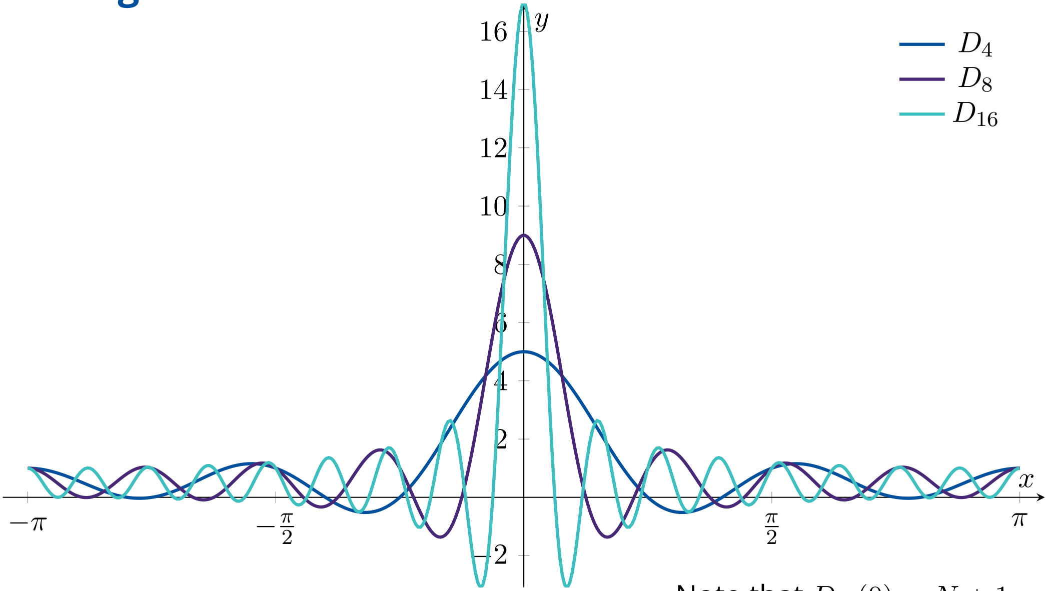

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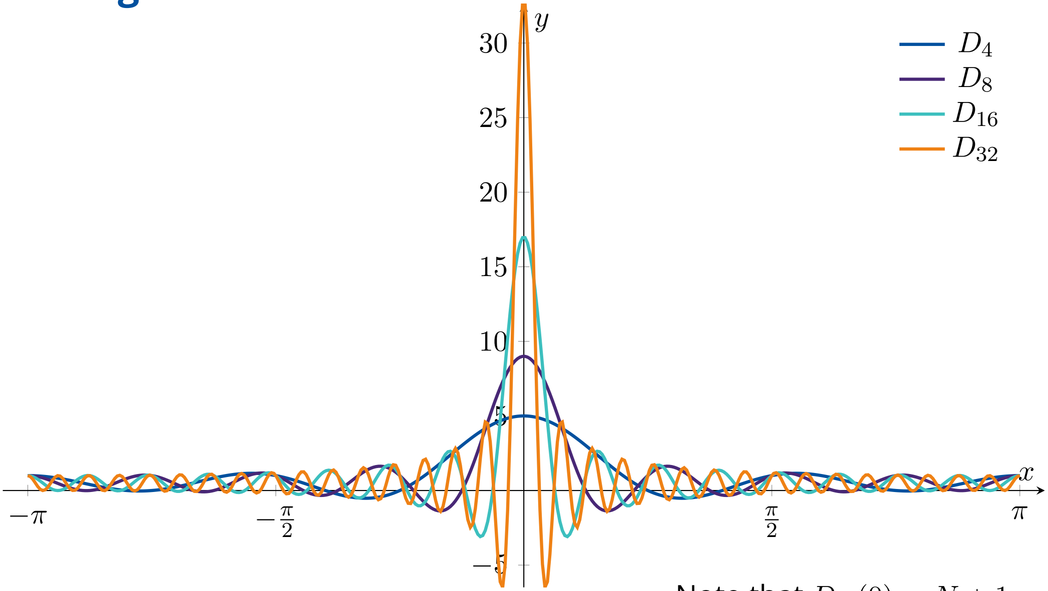

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Convolution

Definition. For two 2π-periodic functions f, g the convolution (norsk.: konvolusjon, dt.: Faltung) is denoted by $f * g$ and defined by

$$(f * g)(x) = \int_{-\pi}^{\pi} f(y)g(x-y)\,\mathrm{d}y.$$

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Lemma. For two 2π-periodic functions it holds that

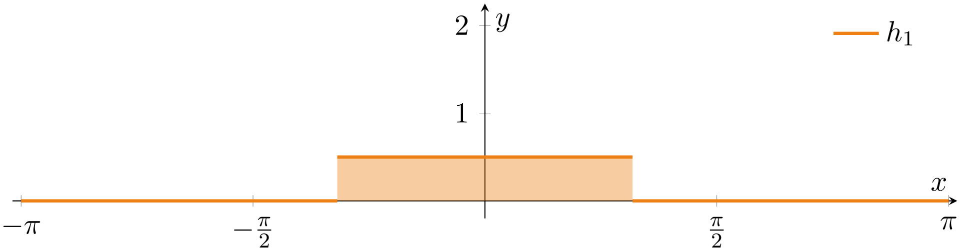
$$(f * g)(x) = \int_{-\pi}^{\pi} f(y)g(x-y)\,\mathrm{d}y. = \int_{-\pi}^{\pi} g(y)f(x-y)\,\mathrm{d}y = (g * f)(x).$$

Proof.

Understanding Convolution – Avergaging

Let's convolve our function f first with a special function to understand convolutions better.
For $0 < \varepsilon < \pi$ let h_ε be the 2π periodic function given by

$$h_\varepsilon(x) = \frac{1}{2\varepsilon} X_{[-\varepsilon,\varepsilon]}(x) = \begin{cases} \frac{1}{2\varepsilon} & \text{for } |x| \leq \varepsilon \\ 0 & \text{else.} \end{cases}$$



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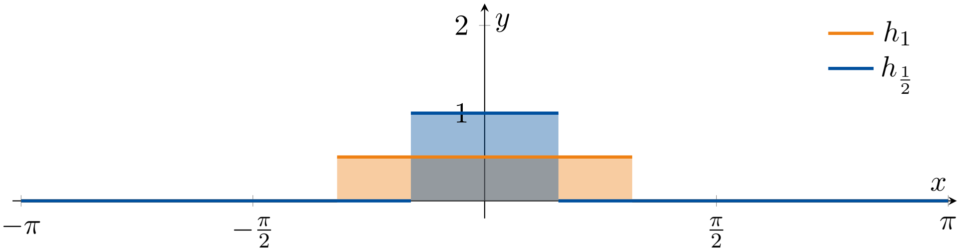

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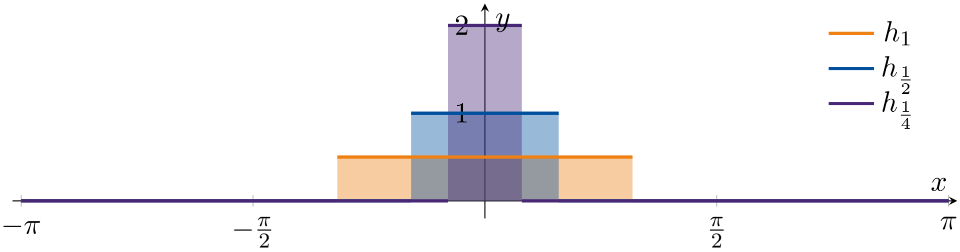

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If f is continuous this follows directly from the fundamental theorem of calculus:

Convolution and Fourier Coefficients

Lemma. Let f, g be 2π-periodic functions, that are absolutely integrable. Then we can relate convolution and Fourier coefficients as

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Note 2. with this Lemma, we can write $S_N f = \tilde{D}_N * f$

Pointwise Convergence of Fourier Series

Theorem.

Let f be a 2π periodic function that is piecewise continuously differentiable such that at a jump the left and right hand side limits

$$f(x^-) := \lim_{y \to x^-} f(x) \qquad \text{and } f(x^+) := \lim_{y \to x^+} f(x)$$

as well as the left and right derivatives at the jumps x

$$\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} \qquad \text{and } \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

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$$\lim_{N\to\infty} S_N f(x) = \frac{1}{2}\big(f(x^-)+f(x^+)\big)$$

At points where f is continuous, we obtain pointwise convergence to f

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Proof. We omit the proof here.

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Fourier Series on $[-L, L]$ – Idea

We can also consider functions of period $2L$, $L > 0$.
Idea. "Stretch/Squeeze" all formulae we had until now.

Example. since $\sin(x)$ is 2π-periodic, $\sin\left(\frac{\pi}{L}x\right)$ is $2L$-periodic.

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Then

$$\left\{\mathrm{e}^{\frac{\mathrm{i}kx\pi}{L}}\right\}_{k\in\mathbb{Z}} \quad \text{and} \quad \left\{\cos\frac{n\pi x}{L}\right\}_{n=0}^{\infty} \cup \left\{\sin\frac{n\pi x}{L}\right\}_{n=1}^{\infty}$$

are orthogonal systems on $[-L, L]$ with respect to the scalar product

$$\langle f, g\rangle = \int_{-L}^{L} f(x)\overline{g(x)}\,\mathrm{d}x$$

Fourier Series on $[-L, L]$

The Fourier series read

$$f \sim \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\frac{\mathrm{i}kx\pi}{L}}, \qquad c_k = \frac{1}{2L} \int_{-L}^{L} f(x) \mathrm{e}^{-\frac{\mathrm{i}kx\pi}{L}} \,\mathsf{d}x$$

and

$$f \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}$$

with coefficients

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \,\mathsf{d}x, \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \,\mathsf{d}x, \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \,\mathsf{d}x.$$

Even and odd extensions

Definition. (nearly a reminder)

A function f on a symmetric interval $I = [-L, L]$, $L > 0$, is called

- is called even if $f(x) = f(-x)$ holds for all $x \in I$.
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These mean that it is enough to know a function on half the interval and we can extend it

Definition. For $f\colon [0, L] \to \mathbb{R}$ we define its

- odd extension $f_{\mathrm{o}}(x) = \begin{cases} f(x) & \text{if } x \in [0, L) \\ -f(-x) & \text{if } x \in (-L, 0) \end{cases}$
- even extension $f_{\mathrm{e}}(x) = \begin{cases} f(x) & \text{if } x \in [0, L) \\ f(-x) & \text{if } x \in (-L, 0) \end{cases}$

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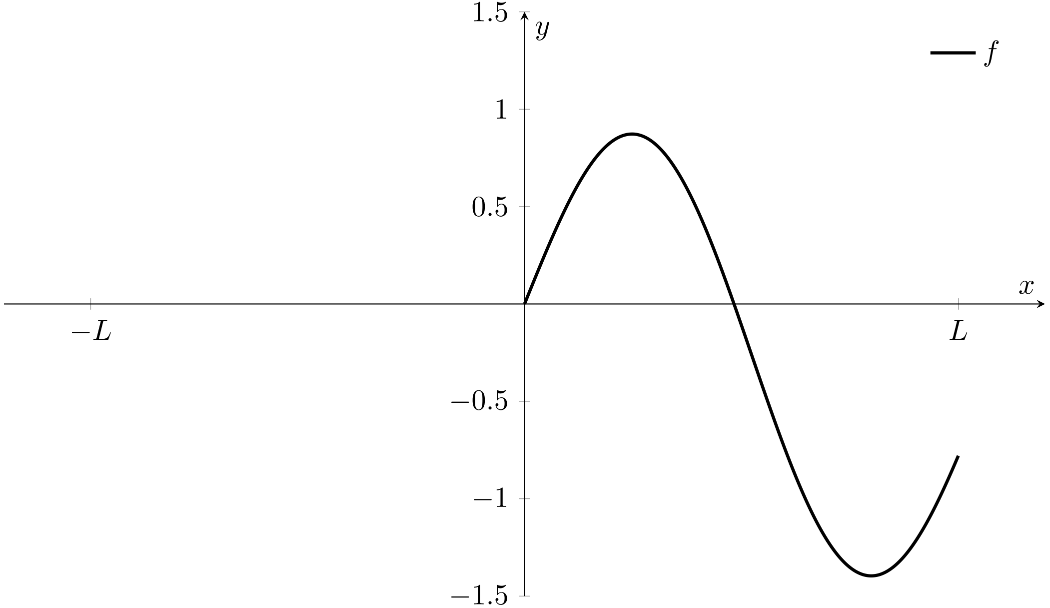
Note. Both f_{o} and f_{e} extend f from $[0, L]$ to $[-L, L]$.

Illustration of even and odd extensions



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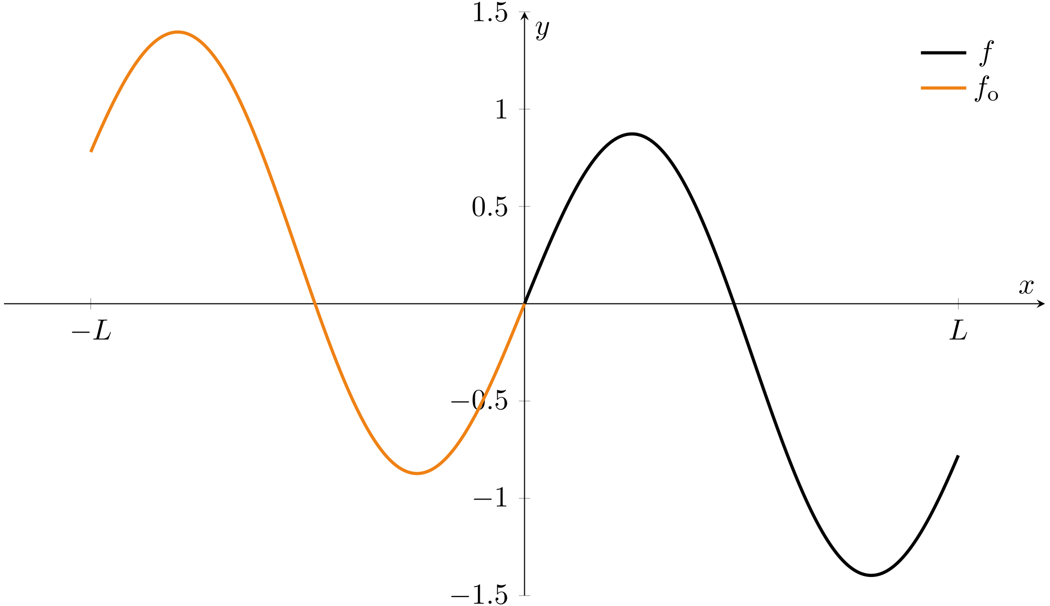

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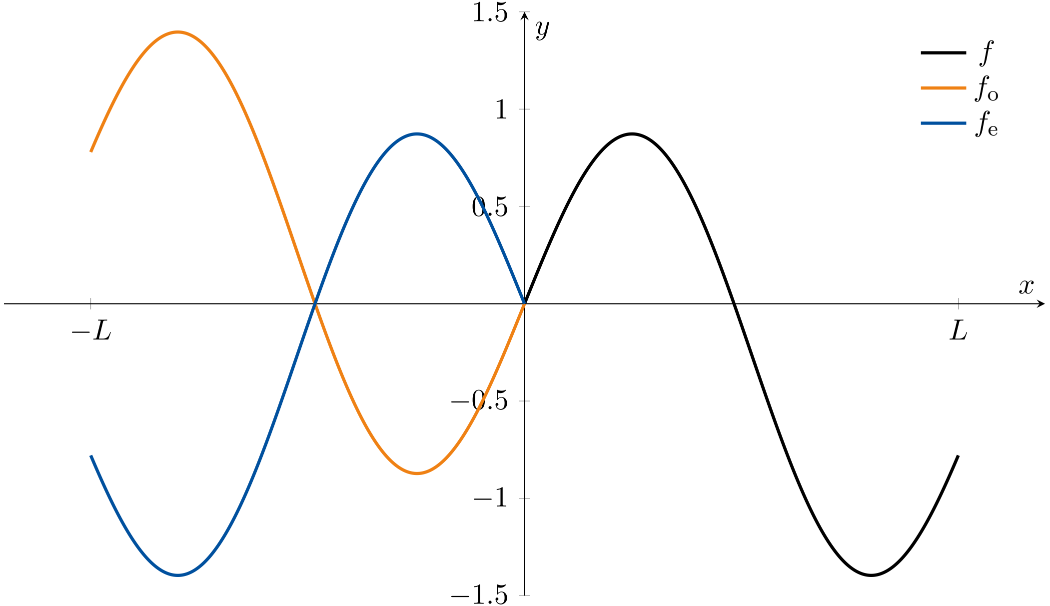

Observation for extensions

Observe that we can compute the Fourier series of f_{e} and f_{o} by just using f.

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For f_e we get

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or in other words a cosine series

Parseval Identity

Theorem.

Let f be given such that $\int_{-L}^{L} |f(x)|^2 \, \mathrm{d}x$ exists and is finite (or similarly: $f \in L^2((-L, L))$). Let

$$f \sim \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\frac{\mathrm{i}kx\pi}{L}}$$

$$\sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}$$

Then

$$\frac{1}{2L} \|f\|^2 = \frac{1}{2L} \int_{-L}^{L} |f(x)|^2 \, \mathrm{d}x = \sum_{k=-\infty}^{\infty} |c_k|^2 = a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

Proof.

Application of Parsevals Identity

For the heaviside function $u(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$ defined on $x \in [-\pi, \pi]$

we can compute its Fourier series

$$u(x) \sim \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\big((2n-1)x\big)$$

and u is smooth in $x = \frac{\pi}{2}$ and satisfies the assumptions of the theorem for pointwise convergence.

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For the heaviside function $u(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$ defined on $x \in [-\pi, \pi]$

we can compute its Fourier series

$$u(x) \sim \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\big((2n-1)x\big)$$

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Note that the sine alternates between ± 1 at the points. Rearranging yields

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{2}\big(1 - \frac{1}{2}\big) = \frac{\pi}{4}$$

Spectrum of periodic functions

For a $2L$-periodic function we defined its Fourier series as

$$f \sim \sum_{k=-\infty}^{\infty} \hat{f}(k) \mathrm{e}^{\mathrm{i}kx\frac{\pi}{L}}$$

where

$$\hat{f}(k) = c_k = \frac{1}{2L} \int_{-L}^{L} f(x) \mathrm{e}^{-\mathrm{i}kx\frac{\pi}{L}} \, \mathrm{d}x \in \mathbb{C}.$$

We can associate with f a sequence of pairs $\left(\hat{f}(k), \frac{k}{2L}\right)_{k\in\mathbb{Z}}$. which is called the spectrum of f.

Amplitude & Phase

Since $\hat{f}(k) = c_k \in \mathbb{C}$ we can also write this complex number as

$$c_k = |c_k| \mathrm{e}^{\mathrm{i}\theta_k},$$

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The number $\frac{1}{2L}$ is called the frequency.

Further properties

If we have two 2π-periodic functions f, g with associated Fourier series

$$f \sim \sum_{k\in\mathbb{Z}} c_k(f)\mathrm{e}^{\mathrm{i}kx} \quad \text{and} \quad g \sim \sum_{k\in\mathbb{Z}} c_k(g)\mathrm{e}^{\mathrm{i}kx}.$$

The the following properties hold

- real functions and $c_k(f)$ if f is real-valued, then $c_k(f) = \overline{c_{-k}(f)}$
- Linearity. the Fourier coefficients of $h_1(x) = \alpha f(x) + \beta g(x)$ ($\alpha, \beta \in \mathbb{R}$) are given by $c_k(h_1) = \alpha c_k(f) + \beta c_k(g)$ for $k \in \mathbb{Z}$.

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- Translation of a function: For $h_2(x) = f(x - x_0)$ for some $x_0 \in [-\pi, \pi)$ yields the Fourier coefficients are $c_k(h_2) = \mathrm{e}^{\mathrm{i}kx_0} c_k(f)$.

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- Differentiation. if f is absolutely continuous and both f, f' are in L_1 (their absolute value ins integrable) then

$$c_k(f') = 2\pi\mathrm{i}k c_k(f)$$