5-14, Let $n, k \in \mathbb{Z}$ be given. Then

$$\langle e^{ik\cdot}, e^{in\cdot} \rangle = \int_{-\pi}^{\pi} e^{ikx}\, \overline{e^{inx}}\, dx$$
$$= \int_{-\pi}^{\pi} e^{ikx} e^{-inx}\, dx$$
$$= \int_{-\pi}^{\pi} \underline{e^{i(k-n)x}}\, dx$$

① If $k=n$

$$\underline{\langle e^{ik\cdot}, e^{ik\cdot} \rangle} = \int_{-\pi}^{\pi} e^{i0x}\, dx = \int_{-\pi}^{\pi} 1\, dx = 2\pi \neq 0$$

② If $k \neq n \Rightarrow k-n \neq 0$

$$\langle e^{ik\cdot}, e^{in\cdot} \rangle = \frac{1}{i(k-n)} e^{i(k-n)x} \Big|_{-\pi}^{\pi}$$
$$= \frac{1}{i(k-n)} \left(e^{i(k-n)\pi} - e^{-i(k-n)\pi} \right)$$
$$= \frac{2}{(k-n)} \left(\frac{e^{i(k-n)\pi} - e^{-i(k-n)\pi}}{2i} \right)$$
$$= \frac{2}{k-n} \sin((k-n)\pi) = 0$$

We have an orthogonal family of functions. $\boxtimes$

5-16, For our function $f = \sum_{\ell=1}^{N} d_\ell \underbrace{\varphi_\ell}_{} \in V_N$ — are orthogonal

we want to compute d_ℓ, $\ell = 1, \dots N$

We look at (for $k \in \{1, \dots, N\}$)

$$\langle f, \varphi_k \rangle = \langle \sum_{\ell=1}^{N} d_\ell \varphi_\ell, \varphi_k \rangle$$

$$\underset{\text{linearity}}{=} \sum_{\ell=1}^{N} d_\ell \underbrace{\langle \varphi_\ell, \varphi_k \rangle}_{= \begin{cases} 0 & \text{if } \ell \neq k \\ \langle \varphi_k, \varphi_k \rangle & \text{if } \ell = k \end{cases}}$$

$$= d_k \langle \varphi_k, \varphi_k \rangle = d_k \underbrace{\|\varphi_k\|^2}_{\neq 0}$$

$$\Rightarrow d_k = \frac{\langle f, \varphi_k \rangle}{\|\varphi_k\|^2}$$

If the $\varphi_1, \dots, \varphi_N$ are even orthonormal then

$$d_k = \langle f, \varphi_k \rangle$$