Math 4N – Fourier Series Lecture III 10/02/2023

5-34, Proof

$$c_k(f*g) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} f(y)\,g(x-y)\,dy\right)e^{-ikx}\,dx$$

$$\overset{\text{Fubini}}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\int_{-\pi}^{\pi} g(x-y)\,dy\; e^{-ikx}\cdot\underbrace{1}_{=e^{-iky}\cdot e^{iky}}\,dx$$

$$\Rightarrow e^{-ikx} = e^{-ik(x-y)}\cdot e^{-iky}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iky}\int_{-\pi\;(x-\pi)}^{\pi\;(x+\pi)} \underbrace{g(x-y)}_{x-y=t}\,e^{-ik\overbrace{(x-y)}^{t}}\,dy\,dx$$

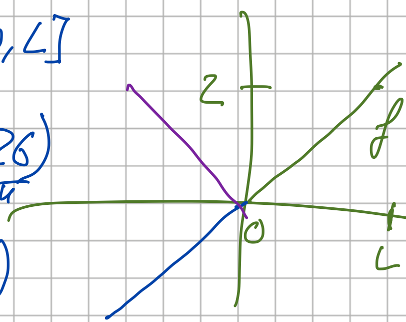
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iky}\underbrace{\int_{-\pi}^{\pi} g(t)\,e^{-ikt}\,dt}_{2\pi c_k(g)}\,dx$$

$$= 2\pi\, c_k(f)\, c_k(g)$$ ☐

5-40, Example $f(x) = x$ on $[0, L]$

$f_e(x) = |x|$ (cf. 5-26, $L=\pi$)

$f_o(x) = x$ (cf. 5-25, $L=\pi$)



5-41, We use the aforementioned "average convergence" (the L_2-convergence) $\lim_{N\to\infty} \|S_N f - f\| = 0$

$\Rightarrow$ This means that $\int_{-L}^{L} |f(x) - S_N f(x)|^2\,dx \longrightarrow 0$ for $N \to \infty$

This is the same as

$$\lim_{N\to\infty} \|S_N f\|^2 = \|f\|^2 = \int_{-L}^{L} |f(x)|^2\,dx < \infty$$

and

$$\|S_N f\|^2 = \int_{-L}^{L} |S_N f(x)|^2 \, dx$$

$$= \int_{-L}^{L} \left(\sum_{k=-N}^{N} c_k(f) e^{ikx\frac{\pi}{L}} \right) \overline{\left(\sum_{\ell=-N}^{N} c_\ell(f) e^{i\ell x\frac{\pi}{L}} \right)} \, dx$$

$$= \sum_{k=-N}^{N} \sum_{\ell=-N}^{N} c_k(f) \overline{c_\ell(f)} \cdot \underbrace{\int_{-L}^{L} e^{\frac{ikx\pi}{L}} e^{-\frac{i\ell x\pi}{L}} \, dx}_{\langle e^{ikx\frac{\pi}{L}}, e^{i\ell x\frac{\pi}{L}} \rangle = \begin{cases} 2L & \text{if } k=\ell \\ 0 & \text{else} \end{cases}}$$

$$= \sum_{k=-N}^{N} c_k(f) \overline{c_k(f)} \cdot 2L$$

$$= 2L \cdot \sum_{k=-N}^{N} |c_k(f)|^2$$

$\blacksquare$

5-43, Spectrum