TMA4125 Plenumsregning 8

Runge-Kutta methods

Butcher tableau

$$
\begin{array}{c|cccc}
c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\
c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\
\vdots & \vdots & & \ddots & \vdots \\
c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\
\hline
 & b_1 & b_2 & \cdots & b_s
\end{array}
$$

differensia...

ifferential equation

$$y'(t) = f(t, y(t)), \quad y(0) = y_0$$

$$y_{n+1} = y_n + h\left(b_1 \cdot k_1 + b_2 \cdot k_2 + \dots + b_s \cdot k_s\right)$$

$$k_i = f\left(t_n + hc_i,\ y_n + h \sum_j a_{ij} \cdot k_j\right)$$

Example 1.

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 2/3 & 2/3 & 0 \\ \hline & 1/4 & 3/4 \end{array}$$

$$y'(t) = \sqrt{t^2 + y(t)^2}, \quad y(0) = e, \quad h = 1$$

a) what is the order of the method?

b) calculate y_1

Solution

a) order 1: $\sum b_i = 1$

$$\sum b_i = \frac{1}{4} + \frac{3}{4} = 1 \quad \text{ok}$$

order 2: ...

order 2: $\sum b_i c_i = 1/2$

$$\sum b_i c_i = \frac{1}{4}\cdot 0 + \frac{3}{4}\cdot\frac{2}{3} = \frac{1}{2} \quad \text{ok}$$

order 3: $\begin{cases} \sum\limits_i b_i \cdot c_i^2 = 1/3 \\ \sum\limits_{i,j} b_i a_{ij} c_j = 1/6 \end{cases}$

$$\sum b_i \cdot c_i^2 = \frac{1}{4}\cdot 0^2 + \frac{3}{4}\cdot\left(\frac{2}{3}\right)^2 = 1/3 \quad \text{ok}$$

$$\sum b_i a_{ij} c_j = \overbrace{\frac{1}{4}\cdot 0\cdot 0}^{i=1,\, j=1} + \overbrace{\frac{1}{4}\cdot 0\cdot\frac{2}{3}}^{i=1,\, j=2}$$

$$+ \underbrace{\frac{3}{4} \cdot \frac{2}{3} \cdot 0}_{i=2,\, j=1} + \underbrace{\frac{3}{4} \cdot 0 \cdot \frac{2}{3}}_{i=2,\, j=2} = 0 \quad \text{not ok}$$

answer: the method has order 2

b) $y'(t) = \sqrt{t^2 + y(t)^2}$, $y(0) = 0$, $h = 1$

$$k_i = f\left(t_0 + h \cdot c_i,\ y_0 + h \cdot \sum_j a_{ij} k_j\right)$$

$f(t, y(0)) = \sqrt{\ldots}$

$$f(t, y(t)) = \sqrt{t^2 + y(t)^2}$$

$$k_1 = f\Big(0 + 1\cdot 0,\ 0 + 1\cdot\big(\underbrace{a_{11}}_{0} k_1 + \underbrace{a_{12}}_{0} k_2\big)\Big)$$

$$= f(0, 0) = 0$$

$$k_2 = f\Big(0 + 1\cdot \frac{2}{3},\ 0 + 1\cdot\big(\underbrace{a_{21}}_{2/3} \underbrace{k_1}_{0} + \underbrace{a_{22}}_{0} k_2\big)\Big)$$

$$= f(2/3, 0) = \sqrt{(2/3)^2 + 0^2} = 2/3$$

$$y_1 = y_0 + h(b_1 k_1 + b_2 k_2) = 0 + 1\Big(\frac{1}{4}\cdot 0 + \frac{3}{4}\cdot\frac{2}{3}\Big)$$

$$= \tfrac{1}{2}$$

Example 2

$$
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
1/3 & 1/3 & 0 & 0 \\
2/3 & 0 & 2/3 & 0 \\
\hline
 & 1/4 & 0 & 3/4
\end{array}
$$

a) order conditions

b) calculate y_1 for same equation

b) calculate y_1 for some

as in example 1.

c) adaptive method

Solution

a)

order 1: $\sum b_i = 1$

$$\sum b_i = \frac{1}{4} + 0 + \frac{3}{4} = 1 \quad \text{ok}$$

order 2: $\sum b_i c_i = 1/2$

$$\sum b_i c_i = \frac{1}{4} \cdot 0 + 0 \cdot \frac{1}{3} + \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2} \quad \text{ok}$$

order 3:

$$\begin{cases} \sum b_i \cdot c_i^2 = 1/3 \\ \sum b_i \, a_{ij} \, c_j = 1/6 \end{cases}$$

$$\sum b_i \cdot c_i^2 = \frac{1}{4} \cdot 0^2 + 0 \cdot \left(\frac{1}{3}\right)^2 + \frac{3}{4} \cdot \left(\frac{2}{3}\right)^2 = 1/3 \quad \text{ok}$$

$$\sum b_i \, a_{ij} \, c_j = \underbrace{\frac{1}{4} \cdot 0 \cdot 0}_{i=1,\ j=1} + \underbrace{\frac{1}{4} \cdot 0 \cdot \frac{1}{3}}_{i=1,\ j=2} + \underbrace{\frac{1}{4} \cdot 0 \cdot 2/3}_{i=1,\ j=3}$$

$$+ \underbrace{0 \cdot \frac{1}{3} \cdot 0}_{i=2,\ j=1} + \underbrace{0 \cdot 0 \cdot \frac{1}{3}}_{i=2,\ j=2} + \underbrace{0 \cdot 0 \cdot 2/3}_{i=2,\ j=3}$$

$$+\underbrace{\frac{3}{4}\cdot 0\cdot 0}_{i=3,\, j=1}+\underbrace{\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{3}}_{i=3,\, j=2}+\underbrace{\frac{3}{4}\cdot 0\cdot\frac{2}{3}}_{i=3,\, j=3}$$

$= 1/6$ ok

order 4: $\sum b_i c_i^3 = 1/4$

$$\sum b_i c_i^3 = \frac{1}{4}\cdot 0^3 + 0\cdot\left(\frac{1}{3}\right)^3 + \frac{3}{4}\cdot\left(\frac{2}{3}\right)^3 = \frac{2}{9}$$ not ok

answer: the method has order 3.

b) $y'(t) = \sqrt{t^2 + y(t)^2}$, $y(0)=0$, $h=1$

$$k_i = f\left(t_0 + h\cdot c_i,\ y_0 + h \sum_j a_{ij} k_j\right)$$

$$y_1 = y_0 + h\left(b_1 k_1 + b_2 k_2 + b_3 k_3\right)$$

Break until 11:10

$$k_1 = f\left(0 + 1\cdot 0,\ 0 + 1\cdot\left(\underbrace{a_{11}}_{0}\cdot k_1 + \underbrace{a_{12}}_{0}\cdot k_2 + \underbrace{a_{13}}_{0} k_3\right)\right)$$

$$= f(0,0) = 0$$

$$k_2 = f(0 + 1\cdot \ldots$$

$$\ldots f\left(\ldots \frac{\ }{3},\ 0 + 1\cdot\left(\underbrace{a_{21}}_{1/3}\cdot \underbrace{k_1}_{0} + \underbrace{a_{22}}_{0} k_2 + \underbrace{a_{23}}_{0} k_3\right)\right)$$

$$= f\left(\frac{1}{3},\ 0\right) = \frac{1}{3}$$

$$k_3 = f\left(0 + 1\cdot\frac{2}{3},\ 0 + 1\cdot\left(\underbrace{a_{31}}_{0}\cdot k_1 + \underbrace{a_{32}}_{2/3}\cdot \underbrace{k_2}_{1/3} + \underbrace{a_{33}}_{0}\cdot k_3\right)\right)$$

$$= f\left(\frac{2}{3},\ \frac{2}{9}\right) = \sqrt{\frac{4}{9} + \frac{4}{81}} = \frac{2}{9}\sqrt{10}$$

$$y_1 = y_0 + h\left(b_1 k_1 + b_2 k_2 + b_3 k_3\right)$$

$$= 0 + 1\cdot\left(\frac{1}{\ }\cdot 0 + 0\cdot\frac{1}{\ } + \frac{3}{\ }\cdot\frac{2}{\ }\cdot\sqrt{10}\right)$$

$$= \frac{\sqrt{10}}{6}$$

c) adaptive methods

method from example 1 has order 2
method from example 2 has order 3

compute error estimate

$$\varepsilon = |\underbrace{y_1}_{\text{from ex 2}} - \underbrace{\tilde{y}_1}_{\text{from ex 1}}|$$

1.2 λ, 7

if $\varepsilon < tol$, keep approximation y_1

if $\varepsilon > tol$, redo the step with a shorter step size h_{new},

$$h_{new} < h \cdot \left(\frac{tol}{\varepsilon} \right)^{p+1}$$

, $p+1$ is the order of the better method

$$y_1 = \frac{\sqrt{10}}{6}, \quad \tilde{y}_1 = 1/2$$

$$\varepsilon = \left| \frac{\sqrt{10}}{6} - 1/2 \right| \approx 0{,}027$$

example 3.

$$
\begin{array}{c|cc}
0 & 0 & 0 \\
2/3 & 2/3 & 0 \\
\hline
 & 1/4 & 3/4
\end{array}
$$

$$y'(t) = \lambda \cdot y(t), \quad y(0) = y_0$$

a) find $R(z)$ s.t. $y_{n+1} = R(h \cdot \lambda) \cdot y_n$

$$y_{n+k} = R(h \cdot \lambda)^k \cdot y_n$$

b) Find the stability region

c) for $\lambda = -2$, find for which h the method is stable

Solution.

a) $y'(t) = \lambda \cdot y(t)$

$f(t, y(t)) = \lambda \cdot y$

$$k_1 = f\big(t_n + h \cdot 0,\ y_n + h(\underbrace{a_{11}}_{0} k_1 + \underbrace{a_{12}}_{0} k_2)\big)$$

$$= f(t_n, y_n) = \lambda \cdot y_n$$

$$K_2 = f\left(t_n + h \cdot \frac{2}{3},\; y_n + h \cdot \left(\underbrace{a_{21}}_{2/3} \cdot \underbrace{k_1}_{\lambda \cdot y_n} + \underbrace{a_{22}}_{0} \cdot k_2\right)\right)$$

$$= f\left(t_n + \frac{2}{3}h,\; y_n + \frac{2}{3} h \lambda y_n\right)$$

$$= \lambda y_n + \frac{2}{3} h \lambda^2 y_n$$

$$y_{n+1} = y_n + h\left(\frac{1}{4} k_1 + \frac{3}{4} k_2\right)$$

$$= y_n + \frac{1}{4} \cdot h \cdot \lambda \cdot y_n + \frac{3}{4} \cdot h \cdot \lambda y_n + \frac{3}{4} \cdot h \cdot \frac{2}{3} \cdot h \cdot \lambda^2 y_n$$

$$= y_n\left(1 + h\lambda + \frac{1}{2}(h\lambda)^2\right)$$

$$R(z) = 1 + z + \frac{1}{2}z^2$$

b)

$$\{ z \in \mathbb{C} : |1 + z + \frac{1}{2}z^2| < 1 \}$$

c) $\lambda = -2$, $z = h\lambda$

$$\frac{1}{2}z^2 + z + 1 = \frac{1}{2}\underbrace{(z+1)^2}_{z^2+2z+1} + \frac{1}{2} \geq 0$$

if $z \in \mathbb{R}$, then $|1 + z + \frac{1}{2}z^2| = 1 + z + \frac{1}{2}z^2$

because $z = h \cdot \lambda = -2h$ and $h \in \mathbb{R}$

$$|1 + z + \tfrac{1}{2} z^2| < 1 \iff 1 - 2h + 2h^2 < 1$$

$$\iff 2h^2 - 2h < 0$$

$$\iff \underbrace{h}_{\geq 0}(2h-2) < 0$$

$$\iff 2h - 2 < 0$$

$$\iff 2h < 2$$

$$\iff h < 1$$

answer: the method is stable for

$$y' = -2y \quad \text{if} \quad 0 < h < 1$$