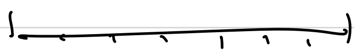
Writing down Discretizations

Consider Initial-Boundary-value problem (IBVP)

$$\begin{cases} u_t - u_{xx} + 3u = f, & x \in [0,1] \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = g(x) \end{cases}$$

a) Write down a seconda order semi-discretization w.r.t. x, with $\Delta x = \frac{1}{N}$

Sol: $x_i = i \cdot \Delta x$



Approximate $u(i\Delta x, t) \approx u_i(t)$, $i = 1, \ldots, N-1$

$$u_t(x_i, t) \approx \frac{d}{dt} u_i(t)$$

$$u_{xx}(x_i, t) \approx \frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{\Delta x^2}, \quad i = 2, \ldots, N-2$$

Let $f_i = f(x_i)$

Introduce "imaginary" points $u_0(t) \approx u(0,t) = 0$

$u_N(t) \approx u(1,t) = 0$

$$u_{xx}(x_1,t) \approx \frac{u_2(t) - 2u_1(t)}{\Delta x^2}$$

$$u_{xx}(x_N,t) \approx \frac{-2u_{N-1}(t) + u_{N-2}(t)}{\Delta x^2}$$

$$\begin{cases} \frac{d}{dt} u_i(t) = \frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{\Delta x^2} - 3u_i(t) + f_i, \quad i=2,\dots,N-2 \\ \frac{d}{dt} u_1(t) = \frac{u_2(t) - 2u_1(t)}{\Delta x^2} - 3u_1(t) + f_1 \\ \frac{d}{dt} u_{N-1}(t) = \frac{-2u_{N-1}(t) + u_{N-2}(t)}{\Delta x^2} - 3u_{N-1}(t) + f_{N-1} \\ u_i(t) = g(x_i), \quad i=1,\dots,N-1 \end{cases}$$

vector $\bar{u}(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_{N-1}(t) \end{pmatrix}$, $\bar{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_{N-1} \end{pmatrix}$

matrix $A = \begin{pmatrix} -\frac{2}{\Delta x^2} - 3 & \frac{1}{\Delta x^2} & & \\ \frac{1}{\Delta x^2} & -\frac{2}{\Delta x^2} - 3 & \frac{1}{\Delta x^2} & \\ & \ddots & \ddots & \ddots \\ & & \frac{1}{\Delta x^2} & -\frac{2}{\Delta x^2} - 3 \end{pmatrix}$

$$\frac{d}{dt}\hat{u}(t) = A\bar{u}(t) + \bar{f}, \quad \bar{u}(0) = \begin{pmatrix} g(x_1) \\ \vdots \\ g(x_{N-1}) \end{pmatrix}$$

b) Discretize in time with Trapezoidal rule and $\Delta t = \frac{1}{M}$

Sol: Recall

$$\boxed{\begin{aligned} &\text{Tr:} \quad y'(t) = F(y(t), t) \\ &\frac{y^{n+1} - y^n}{\Delta t} = \frac{1}{2}\left(F(y^{n+1}, t_{n+1}) + F(y^n, t_n)\right) \end{aligned}}$$

Define $t_n = n \cdot \Delta t$, approximate

$\bar{u}^n \approx \bar{u}(t_n)$

$$\bar{u}^{n+1} = \bar{u}^n + \frac{\Delta t}{2}\left(\left(A\bar{u}^{n+1} + \bar{f}\right) + \left(A\bar{u}^n + \bar{f}\right)\right)$$

$$\bar{u}^0 = \begin{pmatrix} g(x_1) \\ \vdots \\ g(x_{N-1}) \end{pmatrix}$$

c) Set $M = N = \frac{1}{3}$, approximate

$u\left(\frac{2}{3}, \frac{1}{3}\right)$, $g(x) = x(1-x)$, $f = 0$

Sol: $\Delta x = \frac{1}{N} = \frac{1}{3} = \Delta t$

$$u\left(\frac{2}{3}, \frac{1}{3}\right) = u(2\Delta x, 1 \cdot \Delta t) \approx u_2^1$$

$$A = \begin{pmatrix} -\frac{2}{\Delta x^2} - 3 & \frac{1}{\Delta x^2} \\ \frac{1}{\Delta x^2} & -\frac{2}{\Delta x^2} - 3 \end{pmatrix} = \begin{pmatrix} -21 & 9 \\ 9 & -21 \end{pmatrix}$$

$$\bar{u}^0 = \begin{pmatrix} g(x_1) \\ g(x_2) \end{pmatrix} = \begin{pmatrix} x_1(1-x_1) \\ x_2(1-x_2) \end{pmatrix} = \begin{pmatrix} \frac{2}{9} \\ \frac{2}{9} \end{pmatrix}$$

$\vdots$

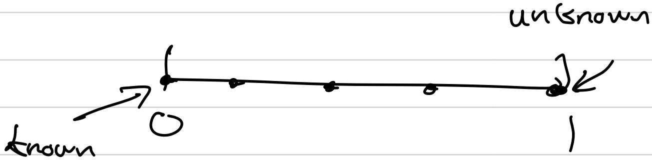
$$\bar{u}^1 = \begin{pmatrix} a \\ b \end{pmatrix}$$

2)

$$\begin{cases} u_t - u_{xx} = 0 \\ u(0,t) = u_x(1,t) = 1 \\ u(x,0) = g(x) \end{cases}$$

a) $u(x_i, t) \approx u_i(t)$, $i = 1, \ldots, N$

$\Delta x = \dfrac{1}{N}$



$$u_{xx}(x_i, t) \approx \frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{\Delta x^2}, \quad i = 2, \ldots, N-1$$

$$\underline{u_{xx}(x_1, t) \approx} \frac{u_2(t) - 2u_1(t) + 1}{\Delta x^2} = \frac{u_2(t) - 2u_1(t)}{\Delta x^2} + \frac{1}{\Delta x^2}$$

$$u_{xx}(1, t) \approx \frac{u_{N+1}(t) - 2u_N(t) + u_{N-1}(t)}{\Delta x^2}$$

What is $u_{N+1}(t)$?

We know $u_x(1,t) = u_x(x_N, t) = 1$

$$u_x(x_N, t) \approx \frac{u_{N+1}(t) - u_{N-1}(t)}{2\Delta x} = 1$$

$$\Leftrightarrow \quad u_{N+1}(t) = u_{N-1}(t) + 2\Delta x$$

$$\frac{\left(u_{N-1}(t) + 2\Delta x\right) - 2u_N(t) + u_{N-1}(t)}{\Delta x^2}$$

$$= \frac{-2u_N(t) + 2u_{N-1}(t)}{\Delta x^2} + \frac{2}{\Delta x}$$

$\vdots$

$$\frac{d}{dt} u_1(t) = \frac{u_2(t) - 2u_1(t)}{\Delta x^2} + \frac{1}{\Delta x^2}$$

$$\frac{d}{dt} u_i(t) = \frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{\Delta x^2}, \quad i = 2, \ldots, N-1$$

$$\frac{d}{dt} u_N(t) = \frac{-2u_N(t) + 2u_{N-1}(t)}{\Delta x^2} + \frac{2}{\Delta x}$$

$$u_i(0) = g(x_i), \quad i = 1, \ldots, N$$