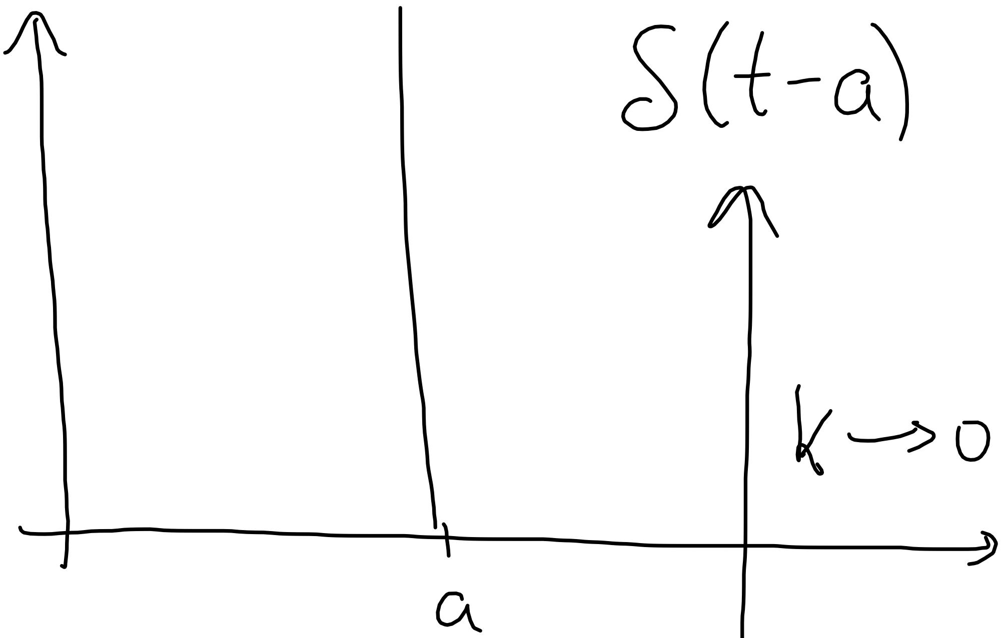
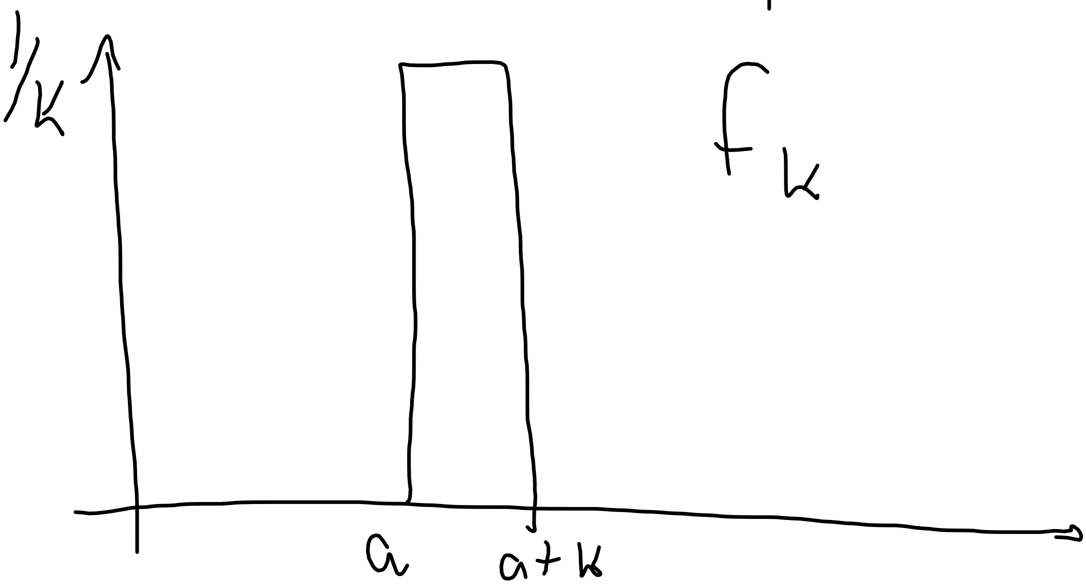
Dirac delta function

$\delta(t-a)$



$k \to 0$

f_k

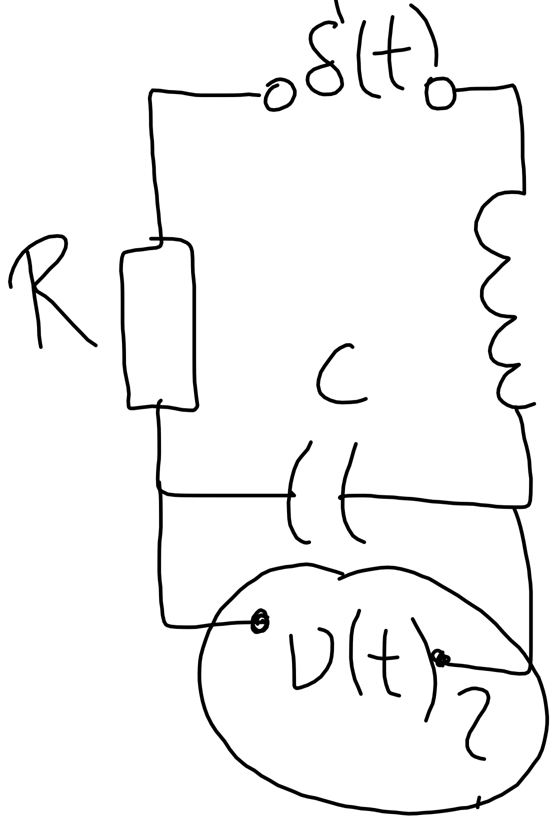


$$\int_0^\infty g(t)\,\delta(t-a)\,dt = g(a)$$

$$\boxed{\mathcal{L}(\delta(t-a))} = \int_0^\infty e^{-st}\,\delta(t-a)\,dt$$

$$= \boxed{e^{-sa}}$$

Example



$R = 20$

$L = 1$

$C = 10^{-4}$

input $= \delta(t)$,, $\delta(t-0)$

No initial Current & Charge

voltage at capacitor

$$= \frac{q(t)}{C}$$ ← charge

We use KVL

$$Li'(t) + Ri(t) + \frac{q(t)}{C} = \delta(t)$$

$$\Rightarrow Lq''(t) + Rq'(t) + \frac{1}{C}q(t) = \delta(t)$$

$$\Rightarrow q''(t) + 20q'(t) + 10000q(t) = \delta(t)$$

Apply LT

$$\underbrace{s^2\mathcal{L}(q) + 20s\mathcal{L}(q)}$$
$$+ 10000\mathcal{L}(q) = e^{0\cdot s} = 1$$

No initial current & charge

$$\Rightarrow \mathcal{L}(q) = \frac{1}{s^2+20s+10000}$$

$$= \frac{1}{(s+10)^2 + (\sqrt{9900})^2}$$

Apply inverse LT

$$\Rightarrow q(t) = \frac{1}{\sqrt{9900}} \sin\left(\sqrt{9900}\, t\right) \underbrace{e^{-10t}}_{\text{1st shifting theorem}}$$

$$\Rightarrow v(t) = \frac{q(t)}{C}.$$

Convolution and integral equations (6.5)

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

The bounds are from 0 to t instead of $-\infty$ to ∞ since f, g are only defined for positive values.

<u>Theorem</u>

$$\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$$

<u>Example</u> Solve the integral equation

$$y(t) - \int_0^t y(\tau)\sin(2(t-\tau))\,d\tau = \sin(2t)$$

This is a convolution:

$$y(t) - y(t) * \sin(2t) = \sin(2t)$$

Apply LT

$$\mathcal{L}(y) - \mathcal{L}(y)\mathcal{L}(\sin(2t)) = \mathcal{L}(\sin(2t))$$

$$\Rightarrow \mathcal{L}(y) = \frac{\mathcal{L}(\sin(2t))}{1 - \mathcal{L}(\sin(2t))}$$

$$= \frac{\frac{2}{s^2+4}}{1-\frac{2}{s^2+4}} = \frac{2}{s^2+2}$$

$$= \sqrt{2} \cdot \frac{\sqrt{2}}{s^2+2}$$

Apply inverse LT

$$\Rightarrow \boxed{y = \sqrt{2} \cdot \sin(\sqrt{2}t)}$$

Application to non-homogeneous linear ODEs

Consider $y'' + ay' + by = r(t)$

with initial conditions:

$y(0) = 0, \; y'(0) = 0$

If we apply LT

$$s^2 \mathcal{L}(y) + as\mathcal{L}(y) + b\mathcal{L}(y) = \mathcal{L}(r)$$

$$\Rightarrow \mathcal{L}(y)(s) = \underbrace{\frac{1}{s^2+as+b}}_{Q(s):\text{ transfer function}} \cdot \mathcal{L}(r)(s)$$

$$= Q(s)\cdot\mathcal{L}(r)(s)$$

$$\Rightarrow \boxed{y(t) = q(t) * r(t)}$$

where $q(t) = \mathcal{L}^{-1}(Q(s))$.

Example: Let's solve the ODE

$$y'' + 3y' + 2y = r(t)$$

$$r(t) = e^{-t}, \quad t \geq 0$$

$$y(0) = 0, \quad y'(0) = 0.$$

Transfer Function

$$Q(s) = \frac{1}{s^2 + 3s + 2}$$

$$= \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$\Rightarrow g(t) = e^{-t} - e^{-2t}$$

$$\Rightarrow y(t) = g(t) * r(t)$$

$$= \int_0^t g(t-\tau) r(\tau)\, d\tau$$

$$= \int_0^t \left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right) e^{-\tau}\, d\tau$$

$$= \boxed{(t-1)e^{-t} + e^{-2t}}.$$

Differentiation and integration of transforms (6.6)

$$\mathcal{L}\left(t f(t)\right)(s) = -\frac{d}{ds}\left(\mathcal{L}(f)\right)(s)$$

$$\mathcal{L}\left(\frac{f(t)}{t}\right)(s) = \int_s^\infty \mathcal{L}(f)(\sigma)\, d\sigma$$

Example: Laguerre's ODE is given by

$$t y'' + (1-t) y' + n y = 0$$

$n \in \mathbb{N}$ $\hookrightarrow$ Non linear

Apply LT

$$\mathcal{L}(t y'') + \mathcal{L}((1-t) y') + n \mathcal{L}(y) = 0$$

$$\Rightarrow \mathcal{L}(t y'') + \mathcal{L}(y') - \mathcal{L}(t y') + n \mathcal{L}(y) = 0$$

$$\Rightarrow -\underline{\frac{d}{ds}\mathcal{L}(y'')} + \underline{\mathcal{L}(y')} + \underline{\frac{d}{ds}\mathcal{L}(y')} + n\mathcal{L}(y) = 0$$

$$\Rightarrow -\underline{\frac{d}{ds}\left(s^2\mathcal{L}(y) - sy(0) - y'(0)\right)}$$

$$+ \underline{s\mathcal{L}(y) - y(0)}$$

$$+ \underline{\frac{d}{ds}\left(s\mathcal{L}(y) - y(0)\right)}$$

$$+ n\mathcal{L}(y) = 0$$

$$\Rightarrow \left[-2s\mathcal{L}(y) - s^2 \frac{d}{ds}\mathcal{L}(y) \cancel{+} y(0) \right]$$

$$+ s\mathcal{L}(y) - y(0)$$

$$+ \left[\mathcal{L}(y) + s\frac{d}{ds}\mathcal{L}(y) \right]$$

$$+ n\mathcal{L}(y) \qquad = 0$$

$$\Rightarrow (s - s^2)\frac{d}{ds}\mathcal{L}(y) + (n+1-s)\mathcal{L}(y) = 0$$

$\hookrightarrow$ Does not depend on the initial conditions

$$Y^{(s)} = \mathcal{L}(y)(s)$$

$$(s-s^2)\,y' + (n+1-s)\,y = 0$$

This is an ODE for y as a function of s.

Separating the variables:

$$y' = \frac{dy}{ds}$$

$$\int \frac{dy}{y} = \int -\frac{n+1-s}{s-s^2}\,ds$$

$$\Rightarrow \ln|Y| = \ln|s-1|^n - \ln|s|^{n+1} + C$$

$$\Rightarrow \boxed{\underset{\mathcal{L}(y)}{\overset{\shortparallel}{Y}} = K\frac{(s-1)^n}{s^{n+1}}}$$

Put $k=1$ for simplification.

Instead of modifying $\mathcal{L}(y)$ to apply inverse LT, we construct a function in the t-world that has $\mathcal{L}(y)$ as its LT.

<u>Construction</u>:

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(t^n e^{-t}) = \frac{n!}{(s+1)^{n+1}}$$

$$\mathcal{L}\left(\frac{d^n}{dt^n}(t^n e^{-t})\right) = \frac{n!\,s^n}{(s+1)^{n+1}}$$

$$\mathcal{L}\left(\frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t})\right) = \frac{(s-1)^n}{s^{n+1}}$$

$$\Rightarrow \boxed{y(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t})}$$

These are called Laguerre polynomials.

$= l_n(t)$

$l_0(t) = 1 \qquad l_1(t) = -t + 1$

$$l_2(t) = \frac{1}{2}(t^2 - 4t + 2)$$