Laplace transforms (6.2–6.6)

$f(t)$, $t \geq 0$

↳ no more than exponential growth

$$\mathcal{L}(f)(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

t-world $\qquad \mathcal{L} \qquad$ **s-world**

ODE $\xrightarrow{\mathcal{L}}$ Algebraic eq.

↓

modified into functions from the tables $\xrightarrow{\mathcal{L}^{-1}}$ ODE

Example: Solve the ODE

$$y'' + 9y = 10e^{-t}$$

$$y(0) = 0 \qquad y'(0) = 0$$

Apply LT

$$\boxed{\mathcal{L}(y'')} + 9\mathcal{L}(y) = 10\,\mathcal{L}(e^{-t})$$

$$\Rightarrow \boxed{s^2\mathcal{L}(y) - s\,\overset{0}{\overset{\parallel}{y'(0)}} - \overset{0}{\overset{\parallel}{y(0)}}} + 9\mathcal{L}(y) = 10\,\mathcal{L}(e^{-t})$$

$$\Rightarrow (s^2+9)\mathcal{L}(y) = \frac{10}{s+1}$$

$$\Rightarrow \mathcal{L}(y) = \underbrace{\frac{10}{(s^2+9)(s+1)}}$$

Use partial fractions to split into things that we know.

$$\frac{10}{(s^2+9)(s+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+9}$$

$$\Rightarrow \frac{10}{(s^2+9)(s+1)} = \frac{A(s^2+9)+(Bs+C)(s+1)}{(s^2+9)(s+1)}$$

$$\Rightarrow 10 = As^2+9A+Bs^2+Bs+Cs+C$$

$$= (A+B)s^2+(B+C)s+(9A+C)$$

$$\Rightarrow \begin{cases} A+B=0 \\ B+C=0 \\ 9A+C=10 \end{cases} \Rightarrow \begin{array}{l} A=1 \\ B=-1 \\ C=1 \end{array}$$

$$\Rightarrow \mathcal{L}(y) = \frac{1}{s+1} + \frac{-s+1}{s^2+9}$$

$$= \frac{1}{s+1} - \frac{s}{s^2+9} + \frac{1}{s^2+9}$$

Apply inverse LT

$$\Rightarrow \boxed{y = e^{-t} - \cos 3t + \frac{1}{3}\sin 3t}$$

(Using tables 6.9
Formula 13 & 14)

Example: (Shifted data problem)

$$y'' - y = 50t - 100$$

$$y(2) = 1 \quad y'(2) = 1$$

We would need $y(0)$ & $y'(0)$ to apply LT, so we make a change of variables

$$\tilde{t} = t - 2 \qquad \tilde{y} = y(\tilde{t})$$

$$\to \tilde{y}'' - \tilde{y} = 50(\tilde{t}+2) - 100$$
$$= 50\tilde{t}$$

$$\tilde{y}(0) = 1 \quad \tilde{y}'(0) = 1$$

Apply LT

$$\mathcal{L}(\tilde{y}'') - \mathcal{L}(\tilde{y}) = 50\mathcal{L}(\tilde{t})$$

$$s^2\mathcal{L}(\tilde{y}) - s\overbrace{\tilde{y}(0)}^{1} - \overbrace{\tilde{y}'(0)}^{1} - \mathcal{L}(\tilde{y}) = \frac{50}{s^2}$$

$$\Rightarrow (s^2-1)\mathcal{L}(\tilde{y}) - s - 1 = \frac{50}{s^2}$$

$$\Rightarrow \mathcal{L}(\tilde{y}) = \underbrace{\frac{s+1}{s^2-1}}_{=\frac{1}{s-1}} + \underbrace{\frac{50}{s^2(s^2-1)}}_{\text{partial fractions}}$$

$$= \frac{1}{s-1} + \frac{50}{s^2-1} - \frac{50}{s^2}$$

Apply inverse LT

$$\Rightarrow \tilde{y} = e^{\tilde{t}} + 50\sinh\tilde{t} - 50\tilde{t}$$

$$\Rightarrow y = e^{t-2} + 50\sinh(t-2) - 50(t-2).$$

Unit step function (6.3)

Definition: The unit step function or Heaviside function is

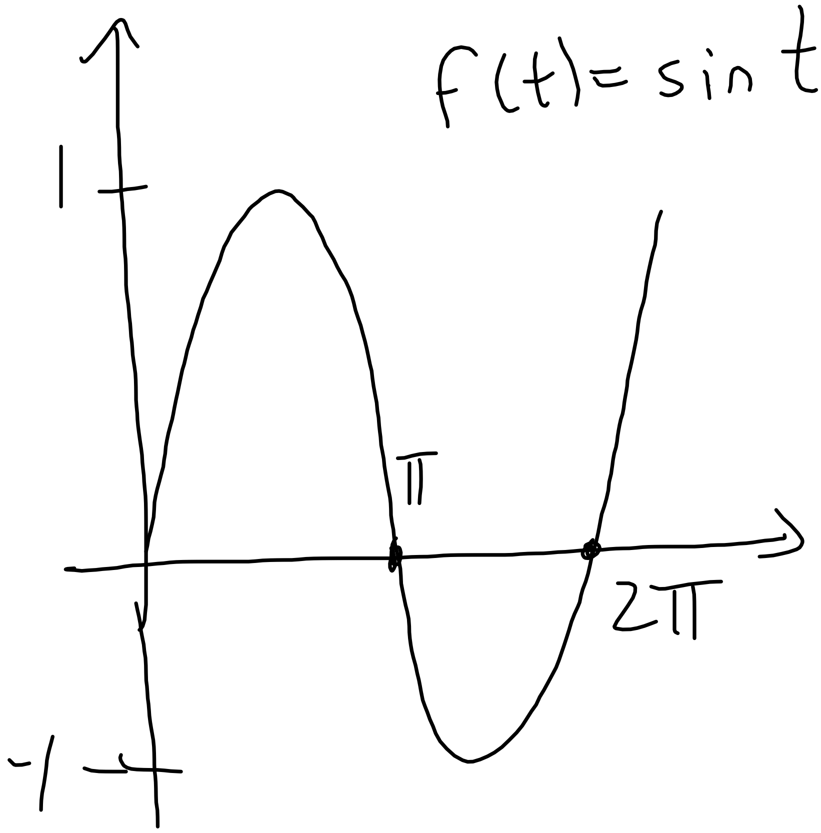
$$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$

$$\mathcal{L}(u(t-a)) = \int_0^\infty e^{-st} u(t-a)\,dt$$

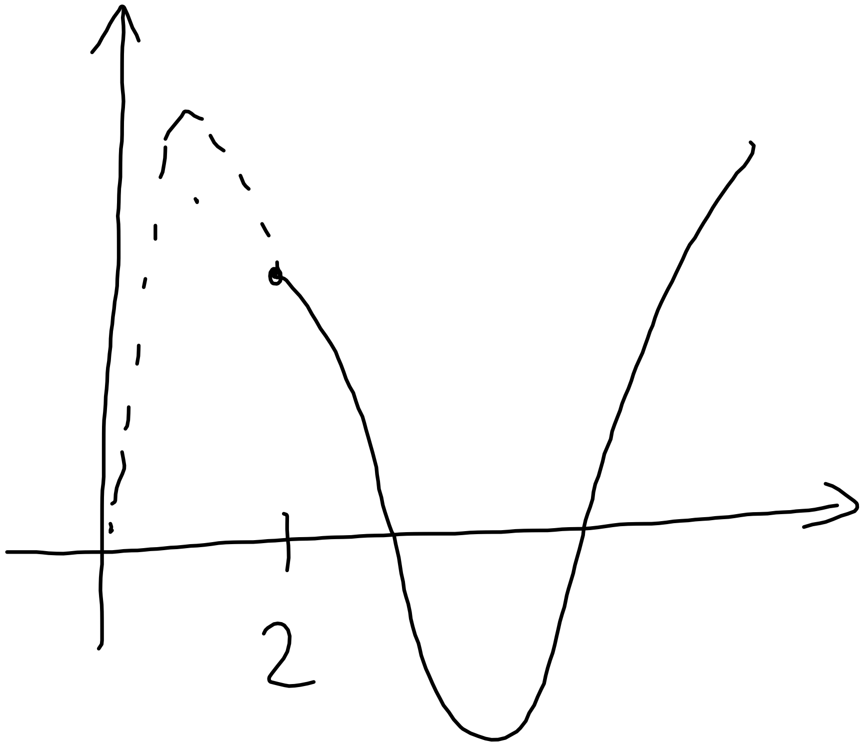
$$= \int_a^\infty e^{-st} \cdot 1 \, dt$$

$$= \boxed{\frac{e^{-as}}{s}}$$

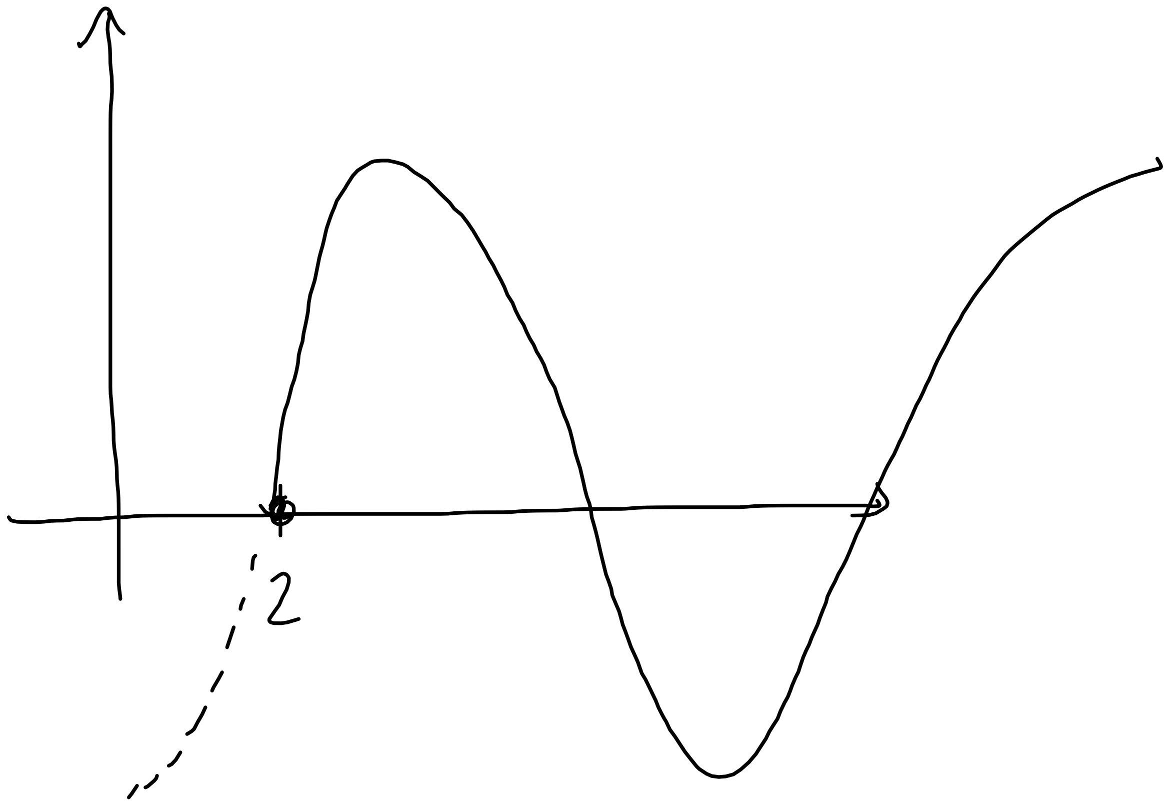
$f(t) = \sin t$



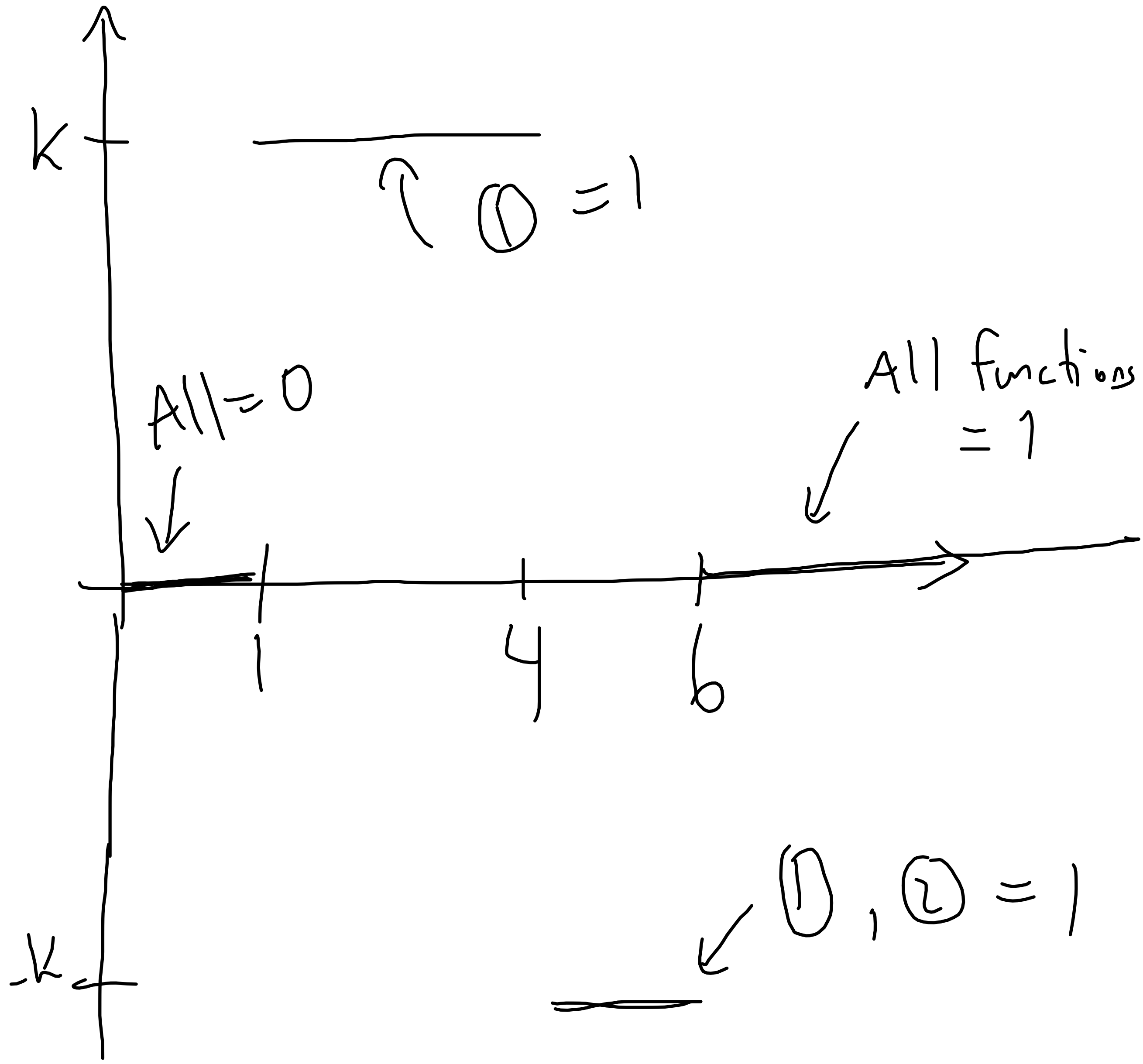
$f_1(t) = \sin(t) \cdot u(t-2)$



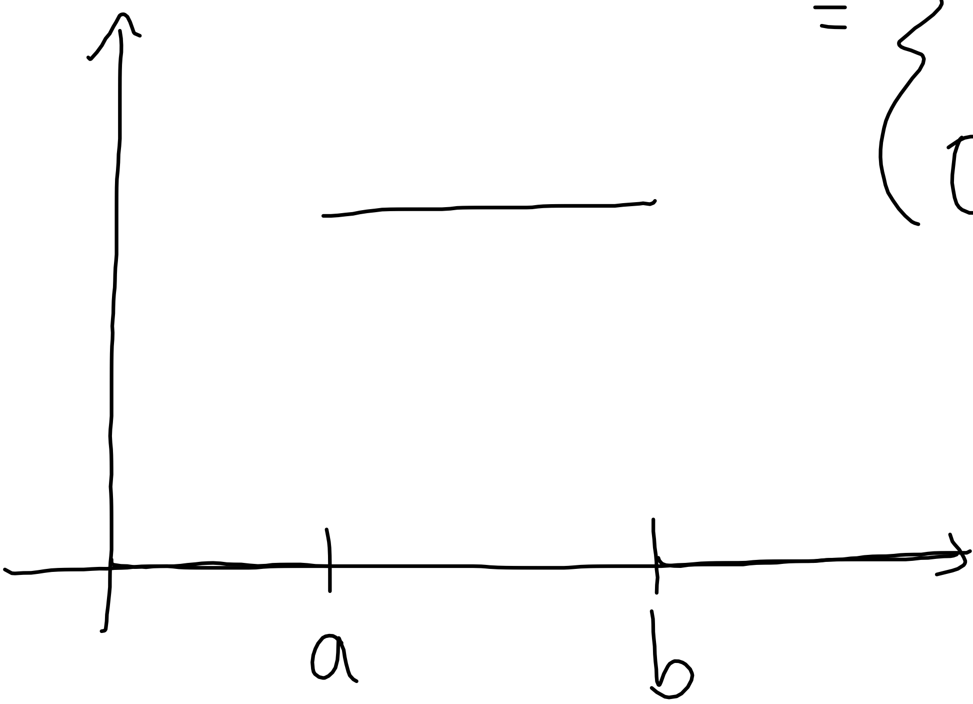
$$f_2(t) = \sin(t-2) \cdot u(t-2)$$



$$k[u(t-1) - 2u(t-4) + u(t-6)]$$



$$u(t-a)-u(t-b) \quad b>a$$



$$= \begin{cases} 1 & a<t<b \\ 0 & \text{else} \end{cases}$$

<u>Theorem</u> (Second shifting theorem)

$$\tilde{f}(t) = f(t-a)u(t-a)$$
$$= \begin{cases} 0 & \text{if } t<a \\ f(t-a) & \text{if } t>a \end{cases}$$

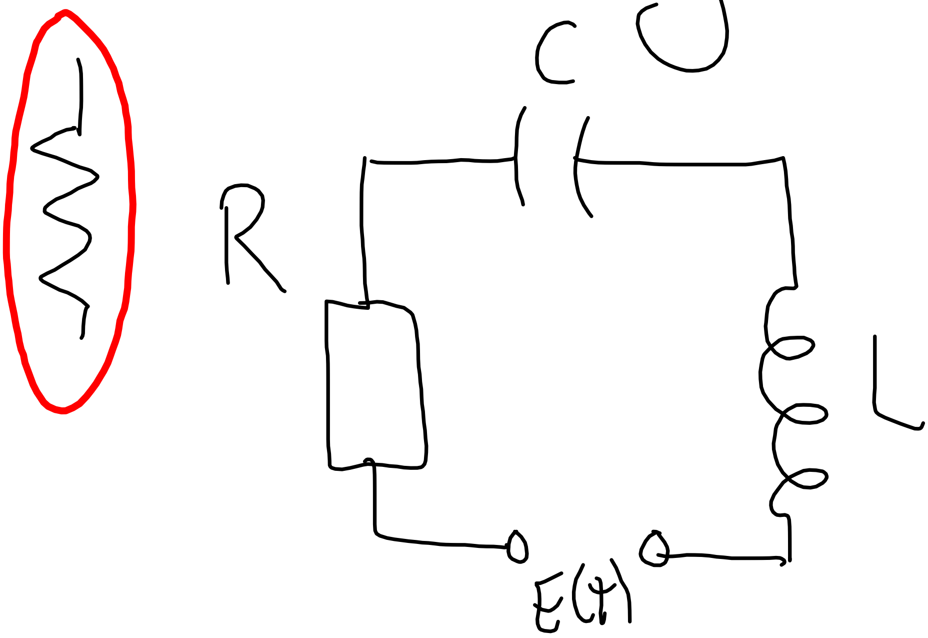
has LT

$$\boxed{e^{-as}\mathcal{L}(f(t))(s).}$$

Also:

$$\boxed{\mathcal{L}(f(t)u(t-a)) = e^{-as}\mathcal{L}(f(t+a))}$$

Modelling circuits



	Units	Voltage drop
Resistor	ohms (Ω)	$Ri(t)$
Inductor	henrys (H)	$Li'(t)$
Capacitor	farads (F)	$q(t)/C$

charge

Kirchoff's Voltage law

$$Li'(t) + Ri(t) + \frac{1}{C}q(t) = E(t)$$

current

$$\Rightarrow \boxed{Li'(t) + Ri(t) + \frac{1}{C}\int_0^t i(\tau)\,d\tau = E(t)}$$

<u>Example</u>: Find the current $i(t)$ in the circuit if the input $E(t)$ is

$$= \begin{cases} 1 & \text{if } 0<t<2 \\ 0 & \text{else} \end{cases}$$

$R=2$, $L=1$ and $C=0.5$ and there is no initial current and charge.
(Initial conditions at 0)

$i(0)=0$

Apply LT to KVL

$$\mathcal{L}\Big(i'(t)\Big)+2\,i(t)+2\int_0^t i(\tau)\,d\tau\Big) \;\; \mathcal{L}\Big(= E(t)\Big)$$

$= 1-u(t-2)$

$$\Rightarrow s\mathcal{L}(i)-\overset{0}{\cancel{i(0)}}+2\mathcal{L}(i)$$

$$+\frac{2}{s}\mathcal{L}(i) \;=\; \frac{1-e^{-2s}}{s}$$

$$\Rightarrow \mathcal{L}(i)=\frac{1-e^{-2s}}{s^2+2s+2}$$

complete the square

$$= \frac{1 - e^{-2s}}{(s+1)^2 + 1}$$

$$= \frac{1}{(s+1)^2+1} - e^{-2s} \cdot \underbrace{\frac{1}{(s+1)^2+1}}$$

This has inv. LT $e^{-t}\sin t$ (First shifting theorem)

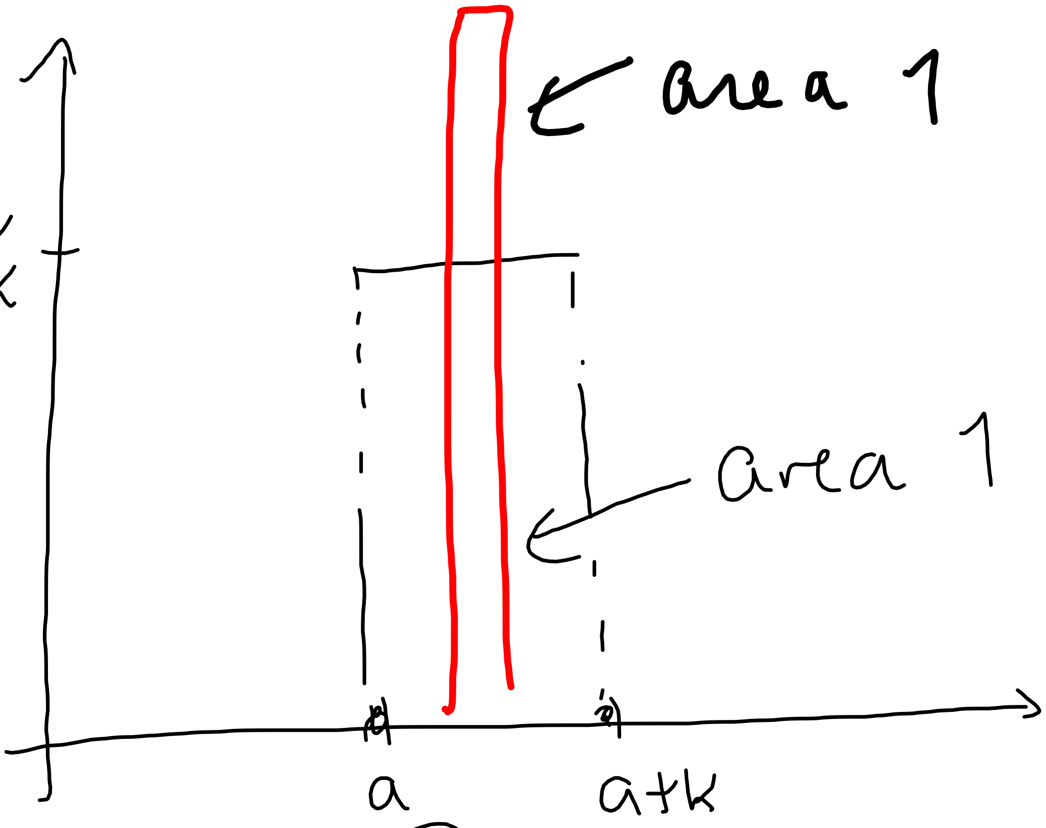
Use second shifting theorem

$$= \boxed{e^{-t}\sin t - \underbrace{e^{-(t-2)}\sin(t-2)u(t-2)}_{\text{only appears after 2 seconds}}}$$

Short impulses and Dirac delta function

Consider the function:

$$f_k(t-a) = \begin{cases} 1/k & \text{if } a \le t \le a+k \\ 0 & \text{else} \end{cases}$$



Impulse lasting k seconds with input $1/k$

Take $k \to 0$.

$\left(1/k \to \infty\right)$

Definition: The Dirac delta function is

$$\delta(t-a) := \lim_{k \to 0} f_k(t-a) = \begin{cases} \infty & \text{if } t=a \\ 0 & \text{else} \end{cases}$$

$$\int_0^\infty f_k(t-a)\,dt = \int_a^{a+k} 1/k \, dt = 1$$

for every $k > 0$.

$$\Longrightarrow \int_0^\infty \delta(t-a)\,dt = 1.$$

In general, if $g(t)$ continuous

$$\int_0^\infty g(t)\delta(t-a)\,dt$$

$$= \int_0^\infty g(t) \lim_{k\to 0} f_k(t-a)\,dt$$

$$= \lim_{k\to 0} \int_0^\infty g(t) f_k(t-a)\,dt$$

$$= \lim_{k \to 0} \int_a^{a+k} g(t) \cdot 1/k \, dt$$

antiderivative

$$= \lim_{k \to 0} \frac{G(a+k) - G(a)}{k}$$

$$= G'(a) = g(a).$$