Laplace transform (chapter 6)

Example: Solve the ODE

$$y' = y$$

Apply FT on both sides

we assume implicitely that $\mathcal{F}(y)$ exists.

$$\mathcal{F}(y') = \mathcal{F}(y)$$

$$\overset{\text{prop. 2}}{\Longrightarrow} i\omega \mathcal{F}(y) = \mathcal{F}(y)$$

$$\underbrace{(1-i\omega)}_{\neq 0}\underbrace{\mathcal{F}(y)}_{=0} = 0$$

$$\Rightarrow \mathcal{F}(y) = 0$$

$$\Rightarrow y = 0.$$

We know that $\underbrace{y=e^x}_{\mathcal{F}(y)\text{ does not exist.}}$ is a solution.

Definition (Laplace transform)

If $f(t)$ is a function defined for all $t \geq 0$, the Laplace transform is a complex-valued function

$$F(s) = \mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$s = \sigma + i\omega$$

$$= \int_0^\infty \underbrace{f(t)\, e^{-i\omega t}}_{\text{Fourier transform}} \underbrace{e^{-\sigma t}}_{\text{new (real)}} \, dt$$

This exists for functions $f(t)$ with exponential growth (somehow cancelled by $e^{-\sigma t}$)

In this course, we will only consider real values of s.

t-world $\qquad$ s-world

$$\begin{tikzcd}[column sep=huge] f(t) \arrow[r, bend left, "\mathcal{L}"] & F(s) \arrow[l, bend left, "\underbrace{\mathcal{L}^{-1}}_{\text{No integral definition}}"] \end{tikzcd}$$

The <u>inverse Laplace transform</u> is the transform such that

$$\mathcal{L}^{-1}(\mathcal{L}(f(t))) = f(t)$$

$$\mathcal{L}(\mathcal{L}^{-1}(F(s))) = F(s)$$

Examples:

a) $\underbrace{f(t) = 1}_{\text{No Fourier transform}}$

$$\mathcal{L}(f(t))(s) = \int_0^\infty f(t)e^{-st}\,dt$$

$$= \int_0^\infty e^{-st}\,dt$$

$$= -\frac{1}{s}e^{-st}\Big]_0^\infty = \frac{1}{s}.$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$$

b) $f(t) = e^{at}$ } No FT , $a \in \mathbb{R}$

$$\mathcal{L}(f(t)) = \int_0^\infty f(t) e^{-st} dt$$

$$= \int_0^\infty e^{at} e^{-st} dt$$

$$= \int_0^\infty e^{-(s-a)t} dt = \frac{1}{s-a}$$

if $s > a$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}.$$

<u>Theorem</u> (Existence and uniqueness of Laplace transform)

If $|f(t)| \leq Me^{kt}$

for some M and k

f piecewise continuous

$\Rightarrow \mathcal{L}(f)(s)$ exists for $s > k$.

The Laplace transform determines uniquely $f(t)$

<u>Properties</u>

1. <u>Linearity</u>:

$$\mathcal{L}(af(t)+bg(t)) = a\mathcal{L}(f(t)) + b\mathcal{L}(g(t))$$

<u>Example</u>:

$$\mathcal{L}(\cosh at) = \mathcal{L}\left(\frac{e^{at}+e^{-at}}{2}\right)$$

$$= \frac{1}{2}\left(\mathcal{L}(e^{at}) + \mathcal{L}(e^{-at})\right)$$

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right)$$

$$= \frac{s}{s^2-a^2} \quad \text{if } s > a.$$

2. Laplace Transform of derivatives

$$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$$

Example

$f(t) = \sin(\omega t)$

$f'(t) = \omega\cos(\omega t)$

$f''(t) = -\omega^2 \sin(\omega t)$

$\Longrightarrow f''(t) = -\omega^2 f(t)$

Also $f(0) = 0 \quad f'(0) = \omega$

Applying LT

$$\mathcal{L}(f'') = -\omega^2 \mathcal{L}(f)$$

$\Rightarrow s^2\mathcal{L}(f) - sf(0) - f'(0) = -\omega^2\mathcal{L}(f)$

$$\Rightarrow s^2\mathcal{L}(f) - \omega = -\omega^2\mathcal{L}(f)$$

$$\Rightarrow \boxed{\mathcal{L}(f) = \frac{\omega}{s^2+\omega^2}}$$

3. <u>LT of integrals</u>

$$\mathcal{L}^{-1}\mathcal{L}\left(\int_0^t \underbrace{f(\tau)}_{\mathcal{L}^{-1}(F(s))} d\tau\right) = \mathcal{L}^{-1}\frac{1}{s}\underbrace{\mathcal{L}(f)}_{F(s)}$$

$$\int_0^t \mathcal{L}^{-1}(F(s))(\tau)\,d\tau = \mathcal{L}^{-1}\left(\frac{1}{s}F(s)\right)$$

Example: Find the inverse LT of $F(s) = \dfrac{2}{s^2 + s/3}$

$$\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{1}{s} \cdot \frac{2}{s+1/3}\right)$$

$$= 2\int_0^t \mathcal{L}^{-1}\left(\frac{1}{s+1/3}\right)(\tau)\, d\tau$$

$$= 2\int_0^t e^{-1/3\,\tau}\, d\tau = \boxed{6 - 6e^{-1/3\,t}}$$

4. Shifting theorem

If $f(t)$ has a $\mathcal{L}T$ for $s>k$ for some k, then for $s>k+a$

$$\mathcal{L}(f(t))(s-a)=\mathcal{L}(e^{at}f(t))(s)$$

Proof:

$$\mathcal{L}(f)(s-a)=\int_0^\infty e^{-(s-a)t}f(t)\,dt$$

$$= \int_0^\infty e^{-st} \cdot \underbrace{e^{at} f(t)}_{g(t)} \, dt$$

$$= \mathcal{L}(g(t))(s) = \mathcal{L}(e^{at} f(t))(s).$$

<u>Example</u>: We know

$$\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}$$

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}$$

Using this, we obtain

$$\mathcal{L}(e^{at}\cos\omega t) = \frac{s-a}{(s-a)^2+\omega^2}$$

$$\mathcal{L}(e^{at}\sin\omega t) = \frac{\omega}{(s-a)^2+\omega^2}$$

Find the inverse LT

same form

$$F(s) = \frac{3s-137}{s^2+2s+401}$$

$$= \frac{3s-137}{\underbrace{s^2+2s+1}_{\text{completing the square}}+400}$$

$$= \frac{3s - 137}{(s+1)^2 + 20^2}$$

$$= \frac{3(s+1) - 140}{(s+1)^2 + 20^2}$$

$$= 3\frac{s+1}{(s+1)^2 + 20^2} - 7\frac{20}{(s+1)^2 + 20^2}$$

Applying inverse $\mathcal{L}T$

$$\mathcal{L}^{-1}(F(s)) = 3e^{-t}\cos(20t) - 7e^{-t}\sin(20t)$$

Solving ODEs using LT (6.2)

Consider an initial value problem (IVP)

input

$$y'' + ay' + by = r(t)$$

output (response)

$$y(0) = K_0 \qquad y'(0) = K_1$$

where a and b are constants.

Strategy:

t-world $\xrightarrow{\mathcal{L}}$ s-world

IVP

get algebraic equation to solve

$\Rightarrow \mathcal{L}(y)$

$\mathcal{L}^{-1}$: modify $\mathcal{L}(y)$ in such a way that we can use the tables

Applying LT on IVP

$$\mathcal{L}(y'' + ay' + by)(s) = \mathcal{L}(r)(s)$$

$$\overset{②}{\Rightarrow} \left[s^2\mathcal{L}(y) - sy(0) - y'(0)\right] + a\left[s\mathcal{L}(y) - y(0)\right] + b\mathcal{L}(y) = \mathcal{L}(r)$$

$$\Rightarrow \mathcal{L}(y) = \frac{(s+a)y(0) + y'(0)}{s^2 + as + b} + \frac{\mathcal{L}(r)}{s^2 + as + b}$$

To find y, need to apply inverse LT.

We call $\frac{1}{s^2+as+b}$ the <u>transfer function</u>, $Q(s)$.

$$\Longrightarrow \mathcal{L}(y) = [(s+a)y(0) + y'(0)]\,Q(s) + \mathcal{L}(r)\,Q(s).$$

If $y(0)=0$, $y'(0)=0$

$\Rightarrow \mathcal{L}(y) = \mathcal{L}(r)\, Q(s)$

$$\Rightarrow Q(s) = \frac{\mathcal{L}(y)}{\mathcal{L}(r)} = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})}$$

Note that Q does not depend on r, nor the initial conditions.

<u>Advantages of the method</u>:

1. No need to solve for the homogeneous solution

and then particular solution.

2. Initial values are taken care of automatically.

3. Will work for some "weird" inputs that have important physical interpretation.