Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w)\, e^{iwx}\, dw$$

$\rightarrow$ Inverse FT of $\hat{f}(w)$

$$\underbrace{\hat{f}(w)}_{\text{Fourier transform}} = 1/\sqrt{2\pi} \int_{-\infty}^{\infty} f(v)\, e^{-ivw}\, d\cancel{w}\, v$$

Inverse Fourier transform

$$\overset{g(w)}{\mathcal{F}^{-1}(g(w))} = 1/\sqrt{2\pi} \int_{-\infty}^{\infty} g(w)\, e^{iwx}\, dw$$

x-world $\mathcal{F}$ ω-world

$$\begin{tikzcd} f(x) \arrow[r, bend left] & \hat{f}(\omega) \arrow[l, bend left, "\mathcal{F}^{-1}"] \end{tikzcd}$$

Mistake in property 3

$$\mathcal{F}\big(xf(x)\big)(\omega) = +i\mathcal{F}'\big(f(x)\big)(\omega)$$

Need f to be abs. integrable

$$\int_{-\infty}^{\infty} |f(x)|\,dx < \infty \quad (\text{very strong})$$

Heat equation: Modeling very long bars (12.7)

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$u(x,t)$: temperature at x and time $t \geq 0$.

bar of infinite length.

No end points $\Rightarrow$

No boundary conditions

Initial condition

$u(x,0) = f(x)$ for all $x \in \mathbb{R}$

Could solve using the method of separation of variables (3 steps).

Instead, use Fourier transforms

We will only obtain solutions where the FT exists.

We apply FT thinking of u as a function of x (t is considered as a constant)

Heat equation: $u_t^{(x,t)} = c^2 u_{xx}^{(x,t)}$

$$\hat{\mathcal{F}}(u_t)^{(w,t)} = c^2 \hat{\mathcal{F}}(u_{xx})^{(w,t)}$$

Cannot use Prop. 2 since the derivative is not with respect to x.

$$\underset{\text{property 2}}{=} c^2 (iw)^2 \hat{\mathcal{F}}(u)$$

$$= -c^2 w^2 \hat{\mathcal{F}}(u)$$

$$\mathcal{F}(u_t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t e^{-i\omega x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{\partial}{\partial t} \int_{-\infty}^{\infty} u e^{-i\omega x}\,dx$$

$$= \frac{\partial}{\partial t} \mathcal{F}(u)$$

$$\Longrightarrow \boxed{\frac{\partial}{\partial t} \mathcal{F}(u)^{(\omega,t)} = -c^2\omega^2 \mathcal{F}(u)^{(\omega,t)}}$$

This is an ODE of $\mathcal{F}(u)$ with respect to t. (ω as a constant)

Solution to ODE

$$\mathcal{F}(u)(\omega,t) = \underbrace{C(\omega)}_{?}\, e^{-c^2\omega^2 t}$$

Using the initial condition

$$C(\omega) = \mathcal{F}(u)(\omega,0)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{u(x,0)}_{\text{initial condition}} e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$$

$$= \hat{f}(\omega)$$

$$\Rightarrow \boxed{\mathcal{F}(u)(w,t) = \hat{f}(w)e^{-c^2w^2t}}$$

To find $u(x,t)$, we apply the inverse FT.

$$\boxed{u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)e^{-c^2w^2t}e^{iwx}\,dw}$$

where

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(v)e^{-iwv}\,dv$$

x world $\xrightarrow{\mathcal{F}}$ ω world

PDE with solution $u(x,t)$ $\longrightarrow$ Using the properties, get a PDE of $\mathcal{F}(u)(\omega,t)$ with respect to t

$\downarrow$ solve

Get a solution for $\mathcal{F}(u)$

$\mathcal{F}^{-1}$

<u>Example</u> Find the solution to the heat equation with initial conditions

$$f(x) = u(x,0) = \begin{cases} |x| & \text{if } -1 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

f is <u>even</u>.

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-c^2\omega^2 t} e^{i\omega x} \, d\omega$$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v)\, e^{-iwv}\, dv$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} |v|\, e^{-iwv}\, dv$$

even $\rightsquigarrow$ cosine integral

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{1} v \cos(wv)\, dv$$

$$\Rightarrow u(x,t) = \frac{2}{2\pi}\int_{-\infty}^{\infty}\int_0^1 v\cos(wv)\,dv\, e^{-c^2w^2t} e^{iwx}\,dw$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos w + w\sin w - 1}{w^2}\, e^{-c^2w^2t} e^{iwx}\,dw$$

Example: Use the FT to solve the PDE

$$t u_x = u_t$$

Find $u(x,t)$

Apply FT on both sides with respect to x.

$$\mathcal{F}(t u_x) = \mathcal{F}(u_t)$$

$$\Rightarrow t\underbrace{\mathcal{F}(u_x)}_{\text{Property (2)}} = \underbrace{\mathcal{F}(u_t)}_{\text{can take } \partial/\partial t \text{ out}}$$

$$\Rightarrow t \cdot i\omega \mathcal{F}(u) = \frac{\partial}{\partial t}\mathcal{F}(u)$$

$\rightsquigarrow$ ODE of $\mathcal{F}(u)$ with respect to t. (ω is a constant)

$$\Rightarrow \mathcal{F}(u)(w,t) = C(w)e^{\frac{t^2}{2}iw}$$

$$C(w) = \mathcal{F}(u)(w,0) = \hat{f}(w).$$

Applying the inverse FT

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)e^{iw\frac{t^2}{2}}e^{iwx}\,dw$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty}\hat{f}(w)e^{iw\left(\frac{t^2}{2}+x\right)}\,dw$$

This is the inverse FT of $\hat{f}$ evaluated at $\frac{t^2}{2}+x$

$$= \boxed{f\left(\frac{t^2}{2}+x\right)}$$

<u>Different form of the solution to the heat equation using convolution</u>

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w)\, e^{-c^2w^2t}\, e^{iwx}\, dw$$

$$= \int_{-\infty}^{\infty} \hat{f}(w)\, \hat{g}(w)\, e^{iwx}\, dw, \text{ where}$$

This is the inverse FT of a product of FTs.

$$\hat{g}(w) = \frac{1}{\sqrt{2\pi}} e^{-c^2 w^2 t}$$

?

$$= (f * g)(x,t)$$

initial condition

$(t$ is a constant$)$

We Find $g(x,t)$.

Already know from previous example

$$\mathcal{F}\left(e^{-ax^2}\right) = \frac{1}{\sqrt{2a}} e^{-\omega^2/4a}$$

$a \neq 0.$

We will modify $\hat{g}(\omega)$ so that it looks like

Choose $a = \dfrac{1}{4c^2 t}$

(to get the same exponant)

$$\Rightarrow \frac{1}{\sqrt{2a}} = \sqrt{2c^2t}$$

$$\hat{g}(w) = \frac{1}{\sqrt{2\pi}} e^{-c^2w^2t}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2c^2t}} \cdot \sqrt{2c^2t}\, e^{-c^2w^2t}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2c^2t}} \cdot \boxed{\frac{1}{\sqrt{2a}} e^{-w^2/4a}}$$

where $a = \dfrac{1}{4c^2t}$.

Applying inverse FT

$$\Rightarrow g(x,t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2c^2t}} e^{-ax^2}$$

$$= \boxed{\frac{1}{2c\sqrt{\pi t}} e^{-x^2/4c^2t}}$$

Using the definition of convolution:

$$u(x,t) = (f * g) = \int_{-\infty}^{\infty} f(p)\, g(x-p)\, dp$$

$$\boxed{= \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(p)\, e^{-\frac{(x-p)^2}{4c^2 t}}\, dp}$$