Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)$$

$$= \sum_{n=-\infty}^{\infty} A_n e^{i\left(\frac{n\pi}{L}x\right)}$$

$$A_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-i\left(\frac{n\pi}{L}x\right)} dx$$

2L - periodic

Fourier integrals

f not necessarily periodic.

$$f(x) = \int_{-\infty}^{\infty} A(\omega) e^{i\omega x} \, d\omega$$

$$A(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(v) e^{-i\omega v} \, dv$$

Example: $f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x \leq 0 \end{cases}$

Computed the Fourier integral:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{i\omega + 1}\, d\omega$$

Use the Fourier integral to compute the value of

$$\int_0^\infty \frac{\cos w + w\sin w}{1+w^2}\,dw$$

We have

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{iwx}}{iw+1}\cdot\frac{1-iw}{1-iw}\,dw$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(1-iw)e^{iwx}}{1+w^2}\,dw$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(1-iw)(\cos wx+i\sin wx)}{1+w^2}\,dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\cos wx + w\sin wx}{1+w^2} + i\left(\frac{\sin wx - w\cos wx}{1+w^2}\right) dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\cos wx + w\sin wx}{1+w^2}\, dw$$

$$+ \underbrace{\frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{\sin wx - w\cos wx}{1+w^2}\, dw}$$

$= 0$ since $f(x)$ is real.

$$= \frac{1}{2\pi} \underbrace{\int_{-\infty}^{\infty} \frac{\cos wx + w\sin wx}{1+w^2}\, dw}_{\text{even}}$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \frac{\cos wx + w\sin wx}{1+w^2}\, dw$$

Choosing $x = 1$

$$\underset{\substack{\| \\ e^{-1}}}{f(1)} = \frac{1}{\pi} \int_{0}^{\infty} \frac{\cos w + w\sin w}{1+w^2}\, dw$$

$$\Longrightarrow \int_{0}^{\infty} \cdots = \pi e^{-1}.$$

Fourier transforms

We call $A(\omega)$ the Fourier transform of f.

$$\mathcal{F}(f)(\omega) \quad , \quad \hat{f}(\omega)$$

Then we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} \, d\omega$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) e^{-i\omega v} \, dv$$

For any function $g(\omega)$ define the <u>inverse Fourier transform</u>

$$\mathcal{F}^{-1}(g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{i\omega x} \, d\omega$$

$$\underline{x \text{ world}} \quad \mathcal{F} \quad \underline{\omega \text{ world}}$$

$$\begin{tikzcd} f(x) \arrow[rr, bend left, "\mathcal{F}"] & & \hat{f}(\omega) \arrow[ll, bend left, "\mathcal{F}^{-1}"] \end{tikzcd}$$

$$\mathcal{F}^{-1}(\mathcal{F}(f))$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(v) e^{-i\omega v} \, dv \right) e^{i\omega x} \, d\omega$$

$$\overset{\text{Yesterday}}{=} f(x)$$

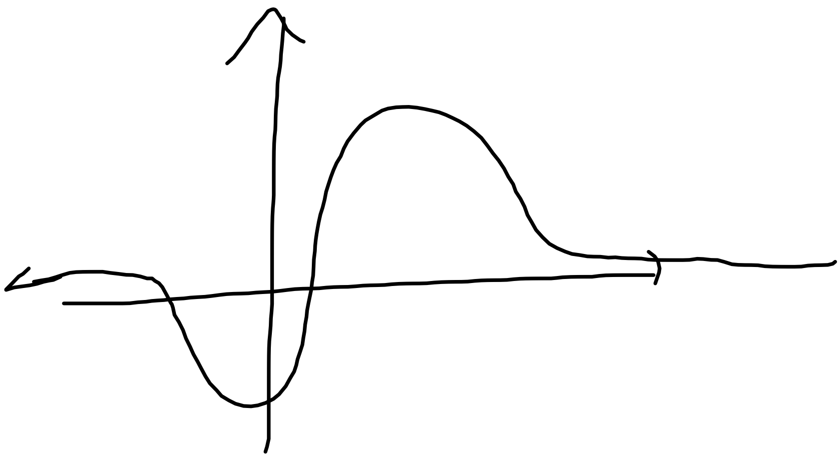
They are inversed of each other.

<u>Theorem</u> (Existence of the Fourier transform)

If $f(x)$ is absolutely integrable $\left(\int_{-\infty}^{\infty} |f(x)|\,dx < \infty \right)$ and piecewise continuous on every finite interval, then $\hat{f}(w)$ exists.

Absolutely integrable

$$\Rightarrow \lim_{\substack{x\to\infty \\ \text{or} \\ x\to-\infty}} f(x) = 0$$



Properties

① **Lineanity**: If f & g are two functions and $a, b \in \mathbb{R}$, then

$$\mathcal{F}(af+bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$$

② If f continuous, f' abs. integrable

$$\mathcal{F}(f')(w) = iw\mathcal{F}(f)(w)$$

<u>Proof</u>: $\mathcal{F}(f') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(v) e^{-i\omega v} \, dv$

Integration by parts

$$u = e^{-i\omega v} \qquad d\alpha = f'(v)\,dv$$
$$du = -i\omega e^{-i\omega v} \qquad \alpha = f$$

$$= \frac{1}{\sqrt{2\pi}} \left(f(x) \underbrace{e^{-i\omega x}}_{\text{bounded}} \Big]_{-\infty}^{\infty} - (-i\omega) \int_{-\infty}^{\infty} f(v) e^{-i\omega v} \, dv \right)$$

$\to 0$ since f is abs. integrable

$= i\omega \mathcal{F}(f)$.

<u>Remark</u>: In general

$$\mathcal{F}(f^{(k)}) = (i\omega)^k \mathcal{F}(f)$$

③ $\mathcal{F}\big(xf(x)\big)(\omega) = -i\mathcal{F}'\big(f(x)\big)(\omega)$

Should be +i

Convolution

The convolution of two functions f, g is defined

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$$

$$= \int_{-\infty}^{\infty} f(x-t)\, g(t)\, dt$$

Convolution theorem

$$\mathcal{F}(f * g) = \sqrt{2\pi}\, \mathcal{F}(f) \cdot \mathcal{F}(g)$$

This implies

$$\mathcal{F}^{-1}\left(\varphi(\omega)\, \psi(\omega)\right) = \mathcal{F}^{-1}(\varphi(\omega)) * \mathcal{F}^{-1}(\psi(\omega))$$

$$\Longrightarrow (f * g)(x) = \underbrace{\int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}(\omega) e^{i\omega x}\, d\omega}_{\text{Inverse FT of the product of the FTs.}}$$

Example: Find the function $f(x)$ such that

$$\int_{-\infty}^{\infty} f(x-t)e^{-2t^2}\,dt = e^{-x^2}$$

for all $x \in \mathbb{R}$.

The integral is a convolution $(f * g)$, where $g(x) = e^{-2x^2}$.

We apply FT on both sides of the equation.

$$\mathcal{F}\left(f(x) * e^{-2x^2}\right) = \mathcal{F}\left(e^{-x^2}\right)$$

$$\Rightarrow \sqrt{2\pi}\,\mathcal{F}(f)\,\mathcal{F}\left(e^{-2x^2}\right) = \mathcal{F}\left(e^{-x^2}\right)$$

convolution theorem.

$$\Rightarrow \mathcal{F}(f) = \frac{\mathcal{F}\left(e^{-x^2}\right)}{\sqrt{2\pi}\,\mathcal{F}\left(e^{-2x^2}\right)}$$

We want to apply the inverse FT on both sides

We first need to simplify the expression on the right.

We compute more generally $\widehat{\ }\big(\underbrace{e^{-b^2x^2}}_{h(x)}\big)$ for some $b \neq 0$.

We use Property ② and compute the FT of a derivative.

$$h'(x) = -2b^2 x e^{-b^2x^2}$$
$$= -2b^2 x h(x)$$

We apply FT on both sides

$$\hat{f}(h'(x)) = -2b^2 \hat{f}(x h(x))$$

We use properties ② and ③

$$i\omega \hat{f}(h)(\omega) = -2b^2 i \hat{f}'(h)(\omega)$$

This is an ODE involving $F(h)$ as a function of ω.

Solution is of the form

$$\hat{f}(h)(\omega) = Ce^{-\omega^2/4b^2}$$

$$C = \mathcal{F}(h)(0)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-b^2 v^2} \cdot \cancel{e^{i \cdot 0 \cdot x}} \, dv$$

$$= \frac{1}{\sqrt{2}\, b}$$ Gaussian integral.

$$\Rightarrow \boxed{\mathcal{F}(h)(\omega) = \mathcal{F}\left(e^{-b^2 x^2}\right)(\omega) = \frac{1}{\sqrt{2}\, b} e^{-\omega^2/4b^2}}$$

Going back to the original problem:

$$\mathcal{F}(f) = \frac{1}{\sqrt{2\pi}} \frac{\mathcal{F}(e^{-x^2})}{\mathcal{F}(e^{-2x^2})}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{\frac{1}{\sqrt{2}} e^{-\omega^2/4}}{\frac{1}{2} e^{-\omega^2/8}}$$

$$= \frac{1}{\sqrt{\pi}} e^{-\omega^2/8}$$

Applying the inverse FT on both sides

$$f(x) = \frac{1}{\sqrt{\pi}} \cdot 2 \cdot \underbrace{\mathcal{F}^{-1}\left(\frac{1}{2} e^{-\omega^2/8}\right)}_{\text{Know} = e^{-2x^2} \text{ (Just found)}}$$

$$\Rightarrow \boxed{f(x) = \frac{2}{\sqrt{\pi}} e^{-2x^2}}$$