# Fourier transform (11.7, 11.9)

## Back to Fourier series:

$f(x)$ has period $2L$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)$$

= sum of sinusoidal waves of frequency

$$\alpha_n = n/2L.$$

Will show $\quad i = \sqrt{-1}$

$$= \sum_{n=-\infty}^{\infty} A_n e^{i\left(\frac{n\pi}{L}x\right)}$$

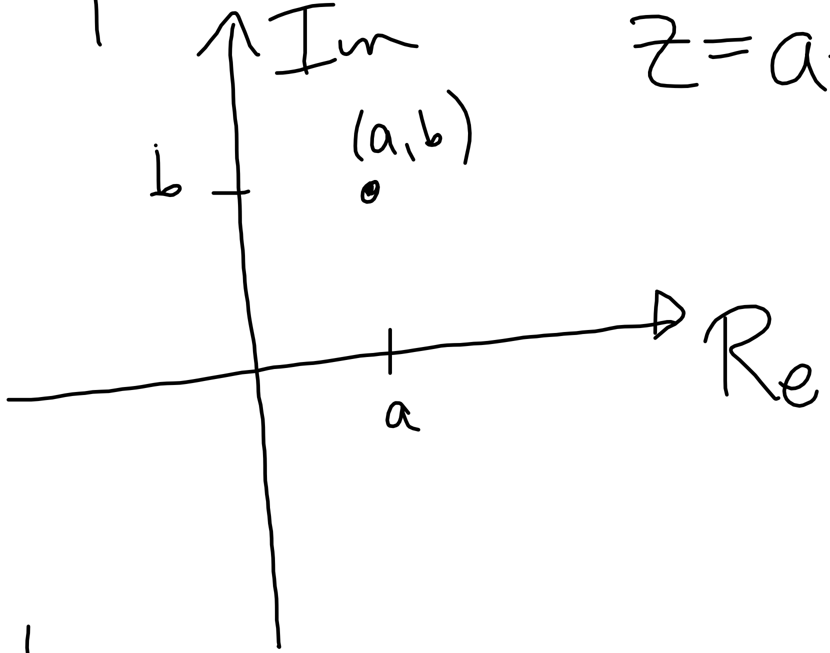
$$= a_0 + \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x + \overbrace{\arctan\left(\frac{a_n}{b_n}\right)}^{\phi_n}\right)$$

$C_n$ = amplitude = $\sqrt{a_n^2 + b_n^2}$

$\phi_n$ = phase shift = $\arctan\left(\frac{a_n}{b_n}\right)$

## Complex numbers

$z = a + ib$




Euler's theorem:

$$e^{i\theta} = \cos(\theta) + i\sin\theta$$

**Facts**: $z = a + ib \Rightarrow \bar{z} = a - ib$

- $\underset{\text{Real part}}{Re}(a+ib) = a = \frac{1}{2}z + \frac{1}{2}\bar{z}$
- $\underset{\text{Imaginary part}}{Im}(a+ib) = b = \frac{1}{2i}z - \frac{1}{2i}\bar{z}$

---

Rewrite the Fourier series,

$$f(x) = a_0 + \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x + \phi_n\right)$$

$$= a_0 + \sum_{n=1}^{\infty} c_n \, \mathrm{Im}\left(e^{i\left(\frac{n\pi}{L}x + \phi_n\right)}\right)$$

$$\underset{\text{Facts}}{=} a_0 + \sum_{n=1}^{\infty} c_n \left( \underbrace{\frac{1}{2i} e^{i\left(\frac{n\pi}{L}x + \phi_n\right)}}_{①} - \underbrace{\frac{1}{2i} e^{i\left(\frac{n\pi}{L}x + \phi_n\right)}}_{②} \right)$$

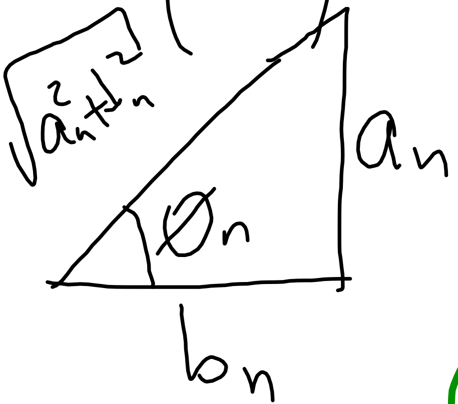
① $\frac{c_n}{2i} e^{i\left(\frac{n\pi}{L}x + \phi_n\right)}$

$$= \frac{c_n}{2i} e^{i\phi_n} \cdot e^{i\left(\frac{n\pi}{L}x\right)}$$

$$= \frac{c_n}{2i} \left( \cos\phi_n + i \sin\phi_n \right) e^{i\left(\frac{n\pi}{L}x\right)} -$$

Recall $\phi_n = \arctan\left(a_n / b_n\right)$

$$\cos(\phi_n) = \cos\left(\arctan\left(a_n / b_n\right)\right)$$




$$= \frac{b_n}{\sqrt{a_n^2+b_n^2}}$$

$$\sin(\phi_n) = \frac{a_n}{\sqrt{a_n^2+b_n^2}}$$

---

$$\rightarrow = \frac{\cancel{c_n}}{2i}\left(\frac{b_n}{\cancel{\sqrt{a_n^2+b_n^2}}} + i\,\frac{a_n}{\cancel{\sqrt{a_n^2+b_n^2}}}\right) e^{i\left(\frac{n\pi}{L}x\right)}$$

Recall $c_n = \sqrt{a_n^2 + b_n^2}$

Also: $\frac{1}{i} = -i$

$$= \boxed{\frac{1}{2}\left(a_n - i b_n\right) e^{i\left(\frac{n\pi}{L}x\right)}}$$

2 : $\frac{c_n}{2i} \overline{e^{i\left(\frac{n\pi}{L}x + \phi_n\right)}}$

$$= \boxed{\frac{1}{2}\left(a_n + i b_n\right) e^{-i\left(\frac{n\pi}{L}x\right)}}$$

$\Rightarrow$ Fourier series

$$= a_0 + \sum_{n=1}^{\infty} \left( \frac{1}{2}(a_n - ib_n) e^{i\left(\frac{n\pi}{L}x\right)} \right) - \left( \frac{1}{2}(a_n + ib_n) e^{-i\left(\frac{n\pi}{L}x\right)} \right)$$

$$= a_0 + \sum_{n=1}^{\infty} \frac{1}{2}(a_n - ib_n) e^{i\left(\frac{n\pi}{L}x\right)} - \underbrace{\left( \sum_{n=1}^{\infty} \frac{1}{2}(a_n + ib_n) e^{-i\left(\frac{n\pi}{L}x\right)} \right)}_{n \rightsquigarrow -n}$$

$$= a_0 + \sum_{n=1}^{\infty} \frac{1}{2}(a_n - i b_n) e^{i\left(\frac{n\pi}{L}x\right)}$$

$$- \sum_{-\infty}^{-1} \frac{1}{2}(a_{-n} + i b_{-n}) e^{i\left(\frac{n\pi}{L}x\right)}$$

$$= \boxed{\sum_{n=-\infty}^{\infty} A_n e^{i\left(\frac{n\pi}{L}x\right)}}$$

$$A_n = \begin{cases} \frac{1}{2}(a_n - i b_n) & n \geq 1 \\ \frac{1}{2}(a_n + i b_n) & n \leq -1 \\ a_0 & n = 0 \end{cases}$$

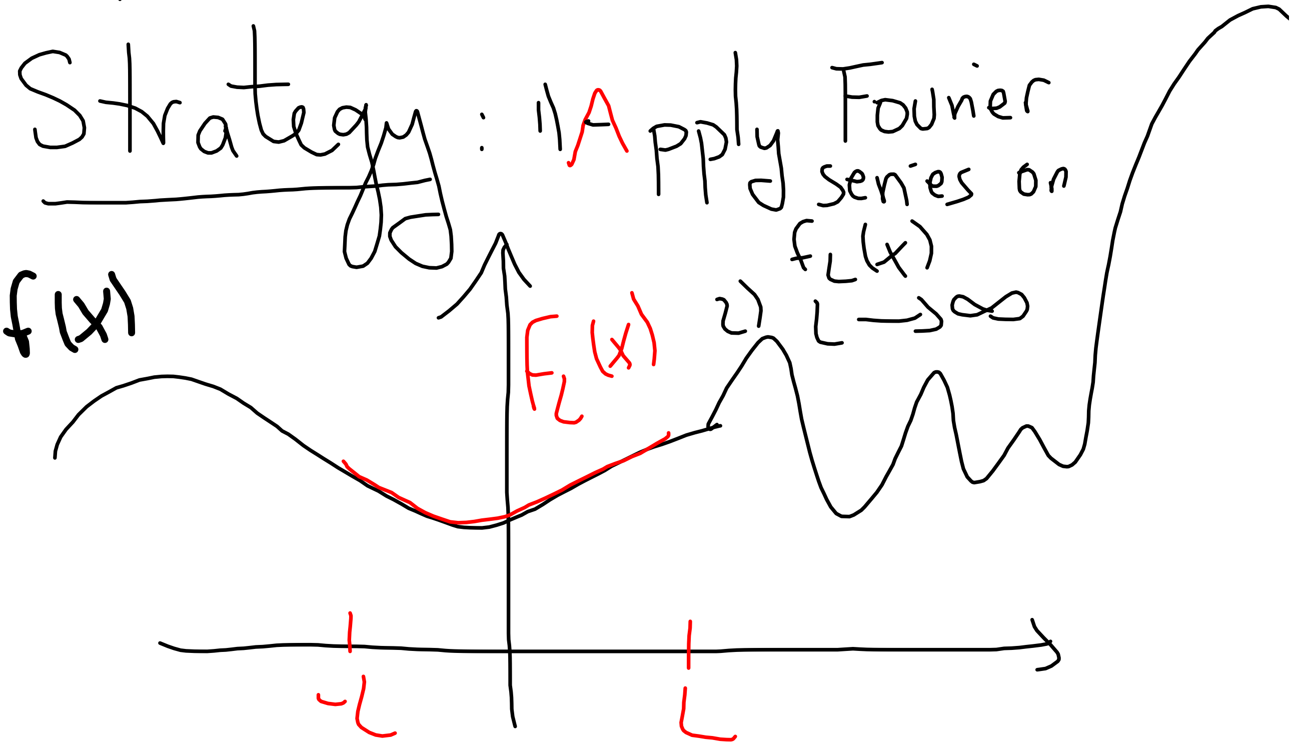
We can compute $A_n$ using Euler formula for $a_n$ & $b_n$

$$\boxed{A_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-i\left(\frac{n\pi}{L}x\right)}\, dx}$$

# Fourier integrals

Extend the notion of Fourier series to non-periodic functions.

**Strategy:** 1) Apply Fourier series on $f_L(x)$
2) $L \to \infty$



Fourier series of $f_L(x)$

$$f_L(x) = \sum_{-\infty}^{\infty} A_n e^{i(\omega_n x)}$$

$$\omega_n = \frac{n\pi}{L} \quad \left(\text{Frequency} = \omega_n / 2\pi\right)$$

Using Euler formula for $A_n$

$$= \sum_{-\infty}^{\infty} \left( \frac{1}{2L} \int_{-L}^{L} f(v) e^{-i\omega_n v} \, dv \right) \cdot e^{i\omega_n x}$$

We will integrate over $\omega$ instead of summing over $\omega_n$

Notice that

$$\Delta \omega_n = \omega_{n+1} - \omega_n$$

$$= \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$$

$$\Longrightarrow \frac{1}{L} = \frac{\Delta \omega_n}{\pi}$$

Hence,

$$f_L(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{i\omega_n x} \int_{-L}^{L} f(v) e^{-i\omega_n v} \, dv \, \Delta \omega$$

Take $L \to \infty$

$\Delta \omega \longrightarrow d\omega$

$$\Longrightarrow \boxed{f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v) e^{-i\omega v} \, dv \, e^{i\omega x} \, d\omega}$$

Can rewrite (Fourier series)

$$f(x) = \int_{-\infty}^{\infty} A(\omega) e^{i\omega x} \, d\omega$$

where

$$A(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(v) e^{-i\omega v} \, dv$$

(Euler formula)

This is called the Fourier complex integral of f.

---

If we had done the same thing with our original description of Fourier series, get

$$f(x) = \int_0^\infty a(w)\cos wx + b(w)\sin wx \, dw$$

$$a(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv \, dv$$

$$b(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv \, dv$$

If $f$ is odd, then all $a(\omega) = 0$, so get a Fourier sine integral:

$$f(x) = \int_0^\infty b(\omega) \sin \omega x \, d\omega$$

$$b(\omega) = \frac{2}{\pi} \int_0^\infty f(v) \sin \omega v \, dv$$

If $f$ is even, $b(\omega) = 0$

Get Fourier cosine series

$$f(x) = \int_0^\infty a(\omega)\cos\omega x \, d\omega$$

$$a(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\cos(\omega v)\, dv.$$

Example: Let $f(x)$

$$f(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Find the Fourier Integral of $f$.

$$f(x) = \int_{-\infty}^{\infty} A(w) e^{iwx}\, dw$$

The coefficients:

$$A(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(v) e^{-iwv}\, dv$$

$$= \frac{1}{2\pi} \int_{0}^{\infty} e^{-v} e^{-iwv}\, dv$$

$$= \frac{1}{2\pi} \int_{0}^{\infty} e^{-(iw+1)v}\, dv$$

$$= \frac{-1}{2\pi(i\omega+1)} \cdot e^{-(i\omega+1)v} \Big]_0^{\infty} -$$

**Note:** $\lim_{v\to\infty} e^{-iv} \neq 0$

$= \lim_{v\to\infty} \cos(-v) + i\sin(-v)$

It is however bounded

$\Rightarrow$ But $\lim_{v\to\infty} e^{-(i\omega+1)v}$

$$= \lim_{v\to\infty} \underbrace{e^{-i\omega v}}_{\text{bounded}} \cdot \underbrace{e^{-v}}_{\to 0} = 0$$

real

$$\rightarrow = \boxed{\frac{1}{2\pi(i\omega+1)} = A(\omega)}$$

Now

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{i\omega+1}\, d\omega}$$