Heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$u(x,t)$: temperature in a bar at position x & time t

Solution for boundary conditions

$u(0,t)=0 \quad u(L,t)=0$

$t \geq 0$

$$u(x,t)=\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)e^{-\lambda_n^2 t}$$

$$\lambda_n = \frac{cn\pi}{L} \quad B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$f(x)=u(x,0)=$ initial temperature

Example:

initial temperature / Initial max temperature $= 100$

$$f(x) = 100 \sin\left(\frac{\pi x}{80}\right)$$

$$\rightsquigarrow u(x,t) = 100 \sin\left(\frac{\pi x}{80}\right) e^{-\lambda_1^2 t}$$

Suppose

$\lambda_1^2 = 0.001785$ (given)

When is the maximum temperature reaching $50°$?

The maximum temperature is reached when sin( ) = 1.
At that point:

$$u(x,t) = 100e^{-0.001705t} \overset{?}{=} 50$$

$$\Rightarrow t = \frac{\ln\frac{1}{2}}{(-0.001705)} = \boxed{388}$$

Say $f(x) = 100\sin\left(\frac{3\pi x}{80}\right)$

Comparing initial conditions:

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{80}\right) \underset{\text{WANT}}{=} 100\sin\left(\frac{3\pi x}{80}\right)$$

$B_3 = 100$

$B_n = 0$, $n \neq 3$

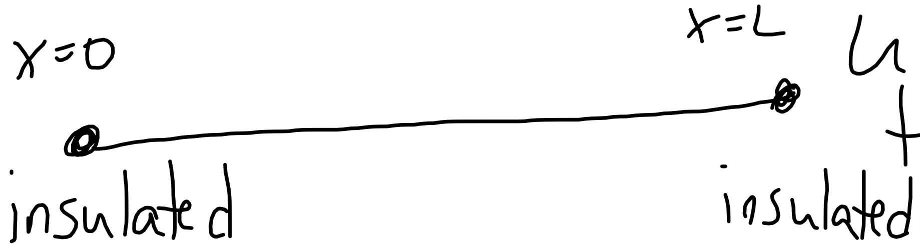
$$\Rightarrow u(x,t) = 100\sin\left(\frac{3\pi x}{80}\right)e^{-\lambda_3^2 t}$$

$$\lambda_3 = \frac{c\cdot 3\cdot \pi}{80} = 3\cdot\frac{c\pi}{80} = 3\cdot\lambda_1$$

$$\Rightarrow \lambda_3^2 = 9\cdot\lambda_1^2$$

$\Rightarrow$ Half life is reached 9 times faster.

Insulated ends



$u(x,t)$ temperature

$\Rightarrow$ no heat flow (rate of change of heat in the x direction) through the end points.

New boundary conditions:

$$u_x(0,t)=0 \quad u_x(L,t)=0$$

$$t \geq 0$$

Same 3 steps

1) Consider $u(x,t) = F(x)G(t)$

$\rightsquigarrow$ 2 ODEs

$$\begin{cases} 1)\ F''(x) - kF(x) = 0 \\ 2)\ \dot{G}(t) - c^2 k G(t) = 0 \end{cases}$$

2) Boundary conditions:

$$0 = u_x(0,t) = \frac{\partial}{\partial x}\left(F(x)G(t)\right)\Big|_{(0,t)}$$

$$= F'(x)G(t)\Big|_{(0,t)} = \underbrace{F'(0)G(t)}_{=0}$$

$$\Rightarrow \boxed{F'(0) = 0}$$

Similarly, $\boxed{F'(L) = 0}$.

Solve eq. 1)

Split into 3 cases:

$$\underbrace{k<0}_{k=-p^2},\ \underbrace{k=0}_{\text{trivial}},\ \underbrace{k>0}_{\text{trivial}}$$

Get

$$F(x) = A\cos(px) + B\sin(px)$$

$$F'(x) = -Ap\sin(px) + Bp\cos(px)$$

Boundary conditions give

$$F'(0) = Bp \underset{\text{WANT}}{=} 0$$

Either $B = 0$ or $p = 0$

If $B = 0$:

$$F'(L) = -Ap\sin(pL) \overset{\text{WANT}}{=} 0$$

If $A=0$: trivial

$p=0$: already one poss.

$\sin(pL)=0$

$\Rightarrow pL=n\pi \Rightarrow p=\frac{n\pi}{L}$

$n=1,2,3,\cdots$

b) If $p=0$: get

$F(x)=A$ a constant term.

Have infinitely many solutions

$$F_n(x) = A_n \cos \frac{n\pi x}{L}$$

$$n = \underset{\uparrow \atop b)}{0},\ \underbrace{1, 2, 3, \ldots}_{a)}$$

Now we solve the second ODE:

Get

$$G_n(t) = e^{-\lambda_n^2 t}$$

$$\lambda_n = \frac{cn\pi}{L}$$

Solutions of the form
$u(x,t) = F(x)G(t)$
are

$$\boxed{\begin{gathered} u_n(x,t) = A_n \cos\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t} \\ n = 0, 1, 2, 3, \ldots \end{gathered}}$$

Note that we have a 0-mode
$u_0(x,t) = A_0 = \text{constant}.$

Step 3: We sum over all previous solutions:

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\lambda_n^2 t}$$

$$= A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\lambda_n^2 t}$$

Remark: $\lim_{t\to\infty} u(x,t) = A_0$

average temperature in the bar as $t\to\infty$.

Using the initial condition:

$u(x,0) = f(x)$

$$\Rightarrow \underbrace{A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)}_{\text{Fourier cosine series}} = f(x)$$

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \text{Average initial temperature}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx \qquad n=1,2,3,\dots$$

Steady 2D heat problems and Laplace's equation

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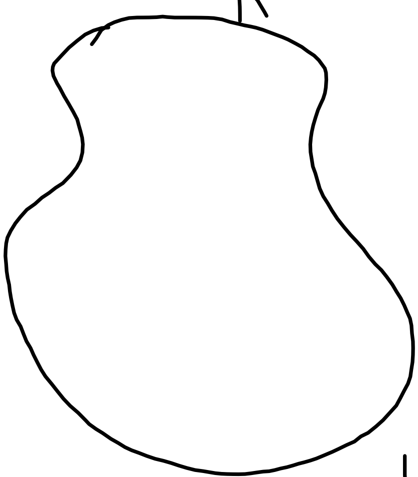


$u(x,y,t)$ temperature at point (x,y) & time $t \geq 0$.

2D Heat equation:

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

<u>Steady solution</u>: A solution that does not depend on time. The steady-state condition exists for cases in which enough time has passed so that u no longer evolves in time.

$$\Longrightarrow \frac{\partial u}{\partial t} = 0$$

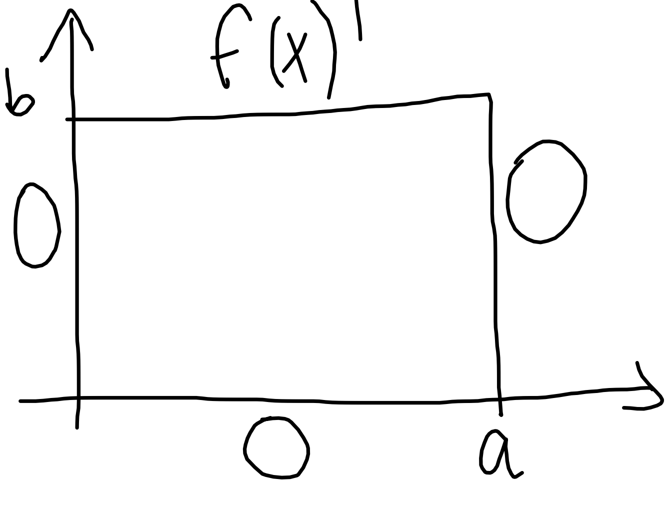
<u>Steady heat equation</u>:

$$0 = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

(Laplace equation)

<u>Example</u>: steady temperature?

$u(x, y)$



Boundary conditions:

$u(x, 0) = 0$

$u(0, y) = 0$

$u(a, y) = 0$

$u(x, b) = f(x)$

Solution: 3 steps

1) $u(x, y) = F(x) G(y)$

Plug this into Laplace equation:

$$\frac{d^2}{dx^2}F(x)G(y)+F(x)\frac{d^2}{dy^2}G(y)=0$$

$$\Rightarrow -\frac{1}{F}\cdot\frac{d^2F}{dx^2}=+\frac{1}{G}\cdot\frac{d^2G}{dy^2}=K$$

Get 2 ODEs

$$\begin{cases} F''(x)+KF=0 & (1) \\ G''(y)-KG=0 & (2) \end{cases}$$

Step 2: From the boundary conditions:

$F(0) = 0$ and $F(a) = 0$

$k = p^2$ (only non-trivial)

Get infinitely many solutions of 1

$F_n(x) = \sin\left(\frac{n\pi}{a} x\right) \quad n = 1, 2, 3, \ldots$

$\left(p = \frac{n\pi}{a}\right)$

ODE ②: $G''(y) - \left(\frac{n\pi}{a}\right)^2 G = 0$

Solutions are of the form

$$G_n(y) = A_n e^{\frac{n\pi y}{a}} + B_n e^{-\frac{n\pi y}{a}}$$

Use the boundary conditions:

$$G_n(0) = 0 \Rightarrow A_n + B_n = 0$$

$$\Rightarrow A_n = -B_n$$

$$\Rightarrow G_n(y) = A_n\left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}\right)$$

$$= 2A_n \sinh\left(\frac{n\pi y}{a}\right)$$

Solutions of the form

$u(x,y) = F(x)G(y)$ $n=1,2,3,\dots$

are $\boxed{u_n(x,y) = A_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)}$

Step 3: Solutions that satisfy $u(x,b) = f(x)$

Sum:

$$u(x,y) = \sum_{n=1}^{\infty}\left(A_n \sinh\left(\frac{n\pi y}{a}\right)\right)\sin\left(\frac{n\pi x}{a}\right)$$

$$u(x,b) = \underbrace{\sum_{n=1}^{\infty} A_n \underbrace{\sinh\left(\frac{n\pi b}{a}\right)}_{\text{number}} \sin\left(\frac{n\pi x}{a}\right)}_{\text{Fourier sine series}} = f(x)$$

<u>Coefficents</u>: $B_n = A_n \sinh\left(\frac{n\pi b}{a}\right)$

$$\underset{\text{Euler}}{=} \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$$

$$\Longrightarrow \boxed{A_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx}$$