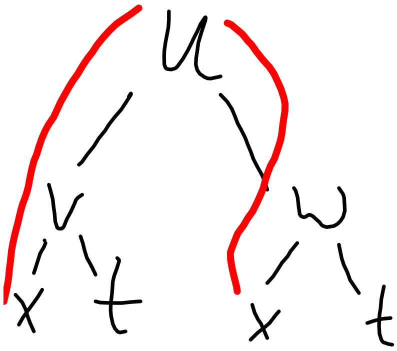
D'Alembert solution

Wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

<u>Strategy</u>: Make a change of variables:

$v = x + ct$ and $w = x - ct$



$\dfrac{\partial^2 u}{\partial x^2}$?

chain rule

From the definition

• $u_x = \dfrac{\partial u}{\partial x} = u_v \underbrace{v_x}_{=1} + u_w \underbrace{w_x}_{=1}$

$= u_v + u_w$

• $u_{xx} = \dfrac{\partial^2 u}{\partial x^2} = (u_v + u_w)_x$

$$= (u_v + u_w)_v \cdot \overset{=1}{v_x}$$

chain rule

$$+ (u_v + u_w)_w \cdot \overset{=1}{w_x}$$

$$= u_{vv} + \underbrace{u_{wv} + u_{vw}} + u_{ww}$$

$$= \boxed{u_{vv} + 2u_{vw} + u_{ww}}$$

if derivatives are continuous

$\frac{\partial^2 u}{\partial t^2}$? Same thing:

$$= \boxed{c^2 (u_{vv} - 2u_{vw} + u_{ww})}$$

Want u to satisfy the wave equation, so want

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\cancel{c^2}(\cancel{u_{vv}} - 2u_{vw} + \cancel{u_{ww}}) = \cancel{c^2}(\cancel{u_{vv}} + 2u_{vw} + \cancel{u_{ww}})$$

$$\Rightarrow \boxed{u_{vw} = 0}$$

This is the wave equation with the change of variables.

$$\frac{\partial^2 u}{\partial w \partial v} = 0$$

We now need to solve this PDE.

We first integrate with respect to w.

$$\Rightarrow \int \frac{\partial^2 u}{\partial w \partial v}\, dw = \int 0 \, dw$$

$$\Rightarrow \frac{\partial u}{\partial v} = h(v)$$

We integrate with respect to v:

$$u = \int h(v)\,dv = \underset{\substack{\| \\ \text{antiderivative} \\ \text{of } h}}{\phi(v)} + \underset{\substack{\| \\ \text{constant}}}{\psi(w)}$$

$$\Rightarrow u(v,w) = \phi(v) + \psi(w)$$

$$\Rightarrow \boxed{u(x,t) = \phi(x+ct) + \psi(x-ct)}$$

where ϕ and ψ are arbitrary functions.

Want a solution that satisfies the initial conditions

(1) $u(x,0) = f(x)$ initial deflection

(2) $u_t(x,0) = g(x)$ initial velocity

$x \in [0, L]$.

$$\boxed{\begin{aligned} (1) &\Rightarrow \phi(x) + \psi(x) = f(x) \\ (2) &\Rightarrow c\phi'(x) - c\psi'(x) = g(x) \end{aligned}}$$

chain rule

② Integrating:

$$c\phi(x) - c\psi(x) = \int_{x_0}^{x} g(s)\,ds + k$$

$$\Rightarrow \boxed{\phi(x) - \psi(x) = \frac{1}{c}\int_{x_0}^{x} g(s)\,ds + K}$$

Combine ① and ②
to isolate $\phi(x)$ & $\psi(x)$

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)\,ds + K$$

$\to x+ct$

$$\psi(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_{x_0}^{x} g(s)\,ds - K$$

$x-ct$

$$\Rightarrow$$

$$u(x,t) = \phi(x+ct) + \psi(x-ct)$$

$$\boxed{= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds}$$

Heat equation (12.6)

1-D: We consider the temperature in a long thin metal bar of constant density and homogeneous material

Temperature $u(x,t)$ at position x & time $t \geq 0$.

The bar is perfectly insulated laterally.
⇒ so heat flows in the x direction only.
No heat is produced or disappears in the body.
$u(x,t)$ satisfies

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Boundary conditions

"The ends are kept at 0"

$u(0,t) = 0 \quad u(L,t) = 0$

for all $t \geq 0$.

Initial condition

$u(x,0) = f(x) \quad x \in [0,L]$

"The initial temperature is given by $f(x)$"

Solution

3-steps:

1) Consider solutions of the form $u(x,t) = F(x)G(t)$.
   Get 2 ODEs.
2) Use the boundary conditions to solve the ODEs.
3) Use the initial conditions.

1) We plug $u(x,t)=F(x)G(t)$ into the heat equation

$$F(x)\dot{G}(t)=c^2F''(x)G(t)$$

$$\Rightarrow c^2\frac{F''(x)}{F(x)}=\frac{\dot{G}(t)}{G(t)}=k$$

$$\Rightarrow \begin{cases} F''(x)-kF(x)=0 \\ \dot{G}(t)-c^2kG(t)=0 \end{cases}$$

$\Rightarrow$

2) Split into 3 cases
$k > 0$, $k = 0$, $k < 0$
to solve the ODEs.
Must satisfy the boundary conditions:
$F(0) = 0$ & $F(L) = 0$
Get solution of the form

$$u_n(x,t) = B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t}$$

$n = 1, 2, 3, \ldots$

where $\lambda_n = \frac{cn\pi}{L}$

3) Use initial conditions:

We sum

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$t=0$

$$= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t}$$

Initial condition:

$u(x,0) = f(x)$

$$\Rightarrow u(x,0) = \underbrace{\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)}_{\text{Fourier sine series for } f} = f(x)$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

(Euler formula)

<u>Notes</u>

- $$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \underbrace{e^{-\lambda_n^2 t}}_{\to 0}$$

  $= 0$ yeah!

- $c^2 = K/\rho\sigma$, where

K is thermal conductivity
σ is the specific heat of the body

ρ density

<u>Example</u>: Find the temperature $u(x,t)$ in a laterally insulated bar which is 80 cm long with initial temperature given by

$$f(x) = 100 \sin\left(\frac{\pi x}{80}\right).$$

and the ends are kept at 0.

General solution

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{80}\, e^{-\lambda_n^2 t}$$

$n=1$

~~$B_n =$ Euler formula~~

Initial conditions: $n=1$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{80} = 100\sin\left(\frac{\pi x}{80}\right)$$

✓

Two Fourier series that are equal must have equal coefficients.

$\Rightarrow n=1: \; B_1 = 100$

$n \neq 1: \; B_n = 0$

coefficients

$$\Rightarrow \boxed{u(x,t) = 100 \sin\left(\frac{\pi x}{80}\right) e^{-\lambda_1^2 t}}$$

$$\lambda_1^2 = \frac{c^2 \pi^2}{80^2}.$$