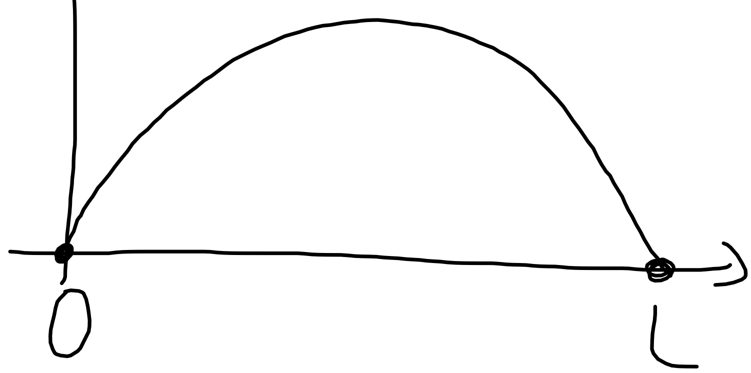
WAVE equation (12.3)

Vibrating string

$u(x,t)$: Deflection at the point x & time t.



$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

<u>Boundary conditions</u>

$u(0,t) = 0 \qquad t \geq 0$

$u(L,t) = 0$

<u>Initial conditions</u>

$u(x,0) = f(x)$

$u_t(x,0) = g(x) \qquad x \in [0,L]$

↳ velocity

<u>Step 1</u>: Consider solutions of the form

$u(x,t) = F(x)G(t)$

Two ODEs:

1) $F''(x) - kF = 0$

2) $\ddot{G}(t) - c^2 kG = 0$

<u>Step 2</u>: Use boundary conditions

Get $F(0)=0$, $F(L)=0$

We solve 1).

<u>Case 1</u>: If $k=0$

$$F''(x)=0$$

$$F(x)=Ax+b$$

Have $F(0)=0 \Rightarrow b=0$

Have $F(L) = 0$

$\Rightarrow AL = 0 \Rightarrow A = 0$

$\Rightarrow F(x) = 0$ trivial solution.

Case 2: $k > 0$ (say $k = \mu^2$)

$$F''(x) - \mu^2 F(x) = 0$$

General solution

$$F(x) = Ae^{\mu x} + Be^{-\mu x}$$

$$F(0) = 0$$

$$\Rightarrow A + B = 0 \Rightarrow A = -B$$

$$F(L) = 0$$

$$\Rightarrow Ae^{\mu L} - Ae^{-\mu L} = 0$$

$$\Rightarrow \underbrace{A}_{=0}\,(\underbrace{e^{\mu L} - e^{-\mu L}}_{\neq 0}) = 0$$

$$\Rightarrow A = 0,\ B = 0$$

trivial solution.

<u>Case 3 : $k < 0$</u> (say $k = -p^2$)

$$F''(x) + p^2 F(x) = 0$$

Characteristic equation:

$$\lambda^2 + p^2 = 0$$

$$\Rightarrow \lambda = \pm ip$$

General solution:

$$F(x) = A\cos(px) + B\sin(px)$$

Boundary conditions:

- $F(0) = 0$

$\Rightarrow A \cdot 1 + B \cdot 0 = 0$

$\Rightarrow A = 0$

- $F(L) = 0$

$$\Rightarrow \underbrace{B}_{\text{Don't WANT}=0} \cdot \underbrace{\sin(pL)}_{=0} = 0$$

$\Rightarrow \sin(pL) = 0$

$$\Rightarrow pL = n\pi$$

$\hookrightarrow$ integer

$$\Rightarrow p = \frac{n\pi}{L}$$

Infinitely many solutions

$$\boxed{F_n(x) = B\sin\left(\frac{n\pi}{L}x\right)}$$

$n = 1, 2, 3, \ldots$

We now solve equation 2) for G with

$$k = -p^2 = -\left(\frac{n\pi}{L}\right)^2.$$

$$\ddot{G}_n + c^2\left(\frac{n\pi}{L}\right)^2 G_n = 0$$

General solution is of the form

$$\boxed{G_n(t) = C_n \cos(\lambda_n t) + C_n^* \sin(\lambda_n t)}$$

where $\lambda_n = cp = \frac{cn\pi}{L}$.

$\Rightarrow$ A general solution of the form $u(x,t)$ is

$$u_n(x,t) = B \sin\left(\frac{n\pi}{L} x\right) \cdot \left(C_n \cos\left(\frac{cn\pi}{L} t\right) + C_n^* \sin\left(\frac{cn\pi}{L} t\right) \right)$$

$$= \left(B_n \cos\left(\frac{cn\pi}{L} t \right) + B_n^* \sin\left(\frac{cn\pi}{L} t \right) \right) \cdot \sin\left(\frac{n\pi}{L} x \right)$$

$$B_n = B \cdot C_n \,, \quad B_n^* = B \cdot C_n^*$$

$$n = 1, 2, 3, 4, \ldots$$

These functions $u_n(x,t)$ are called Eigenfunctions.

The values $\lambda_n = \frac{cn\pi}{L}$ are

Called eigenvalues.
The set $\{\lambda_1, \lambda_2, \ldots\}$
is called the spectrum.

Each u_n represents
a harmonic motion
of frequency $\lambda_n / 2\pi$.

This motion is called
the n^{th} normal mode
of the string.

The function u_1 is called the <u>fundamental mode</u>.

Since $\sin \frac{n\pi}{L} x = 0$

for $x = \frac{L}{n}, \frac{2L}{n}, \ldots, \frac{n-1}{n} L$

we see that the n^{th} normal mode has $n-1$ nodes, that is, points where the string does not move.

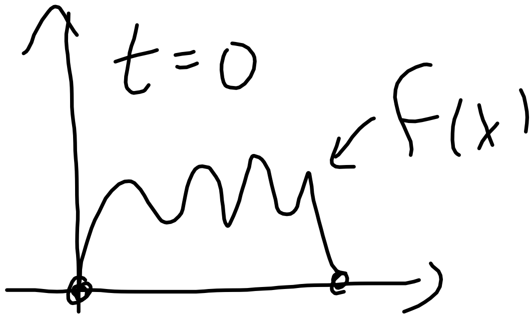
<u>Step 3</u> : General solution that satisfies the initial conditions. In general, a single normal mode does not. The strategy is to sum over all the modes.

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} \left(B_n \cos(\lambda_n t) + B_n^* \sin(\lambda_n t) \right) \cdot \sin\left(\frac{n\pi}{L} x\right)$$

We are looking for the amplitudes B_n & B_n^* such that $u(x,t)$ satisfies the initial conditions.

$$u(x,0) = f(x)$$



$$u(x,0) = \underbrace{\sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi}{L}x\right) \overset{\text{WANT}}{=} f(x)}_{\text{Fourier sine series of } f(x).}$$

$$\Rightarrow B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx \qquad n = 1, 2, \ldots$$

$$u_t(x,0) = g(x)$$

$$\Rightarrow \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

Recall,

$$u(x,t) = \sum_{n=1}^{\infty}\left(B_n \cos\lambda_n t + B_n^* \sin\lambda_n t\right)\sin\left(\frac{n\pi}{L}x\right)$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\left(-B_n \cdot \lambda_n \sin(\lambda_n t) + B_n^* \lambda_n \cos(\lambda_n t)\right)\sin\left(\frac{n\pi}{L}x\right)$$

$$\underbrace{\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} B_n^* \lambda_n \sin\left(\frac{n\pi}{L}x\right) \overset{\text{WANT}}{=} g(x)}_{\text{Fourier sine series}}$$

$$\Rightarrow B_n^* = \frac{2}{\lambda_n \cdot L} \int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

$$= \frac{2}{c \cdot n \cdot \pi} \int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

Special case $g(x) = 0$

(no initial velocity)

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos(\lambda_n t) \cdot \sin\left(\frac{n\pi}{L}x\right)$$

Trigonometric identity:

$$\cos\left(\frac{cn\pi}{L}t\right)\sin\left(\left(\frac{n\pi}{L}\right)x\right)$$

$$= \frac{1}{2}\left[\sin\left\{\frac{n\pi}{L}(x-ct)\right\} + \sin\left\{\frac{n\pi}{L}(x+ct)\right\}\right]$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \frac{B_n}{2}$$

Already know $\displaystyle\sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x$

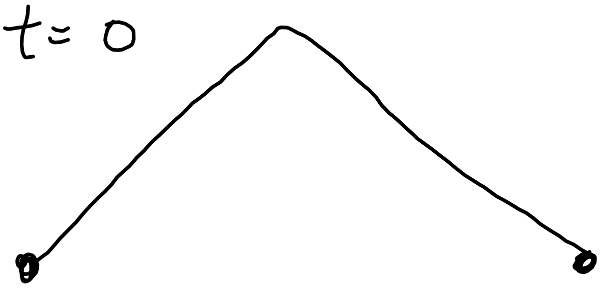
$= f(x)$

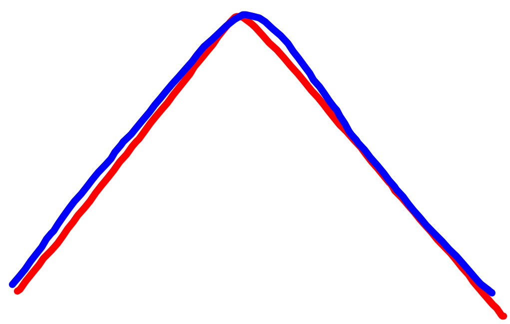
$$\Rightarrow \boxed{u(x,t) = \frac{1}{2}\left[f^*(x-ct) + f^*(x+ct)\right]}$$

where f^* is the odd periodic expansion of f.

Example:

Initial deflection
$t = 0$





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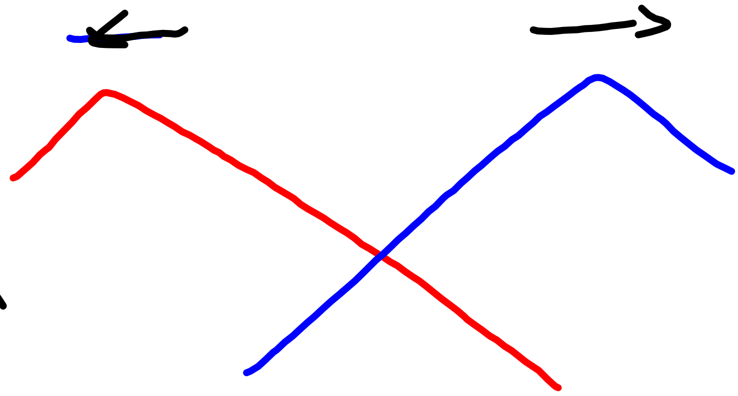

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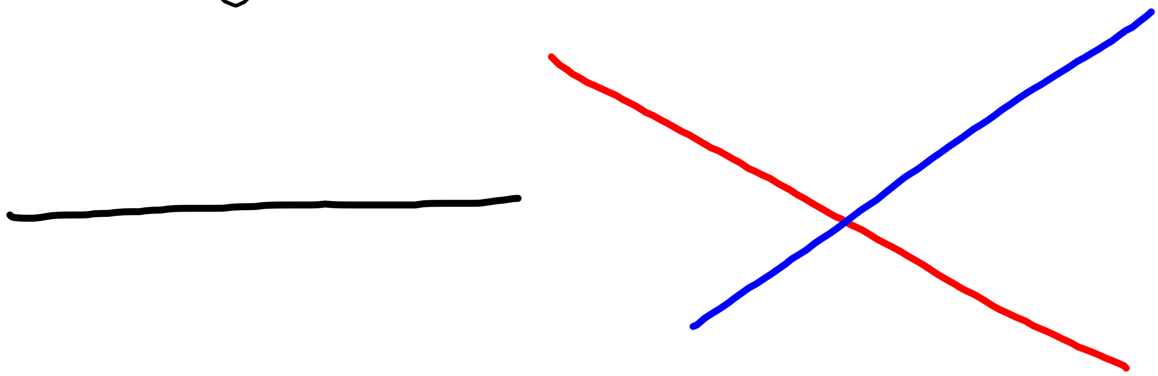
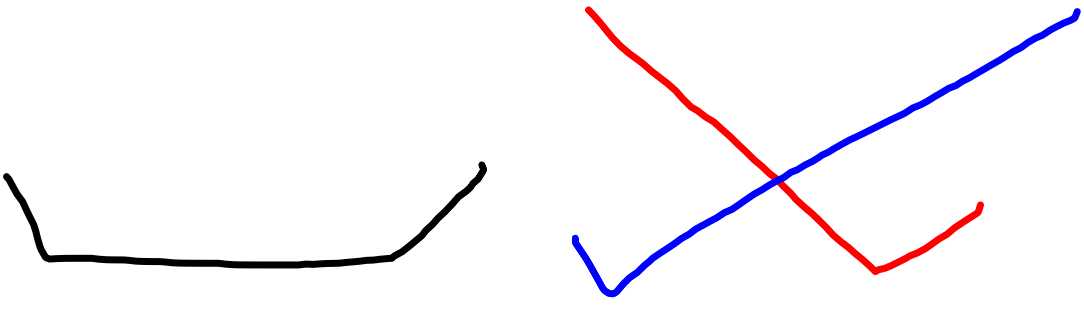
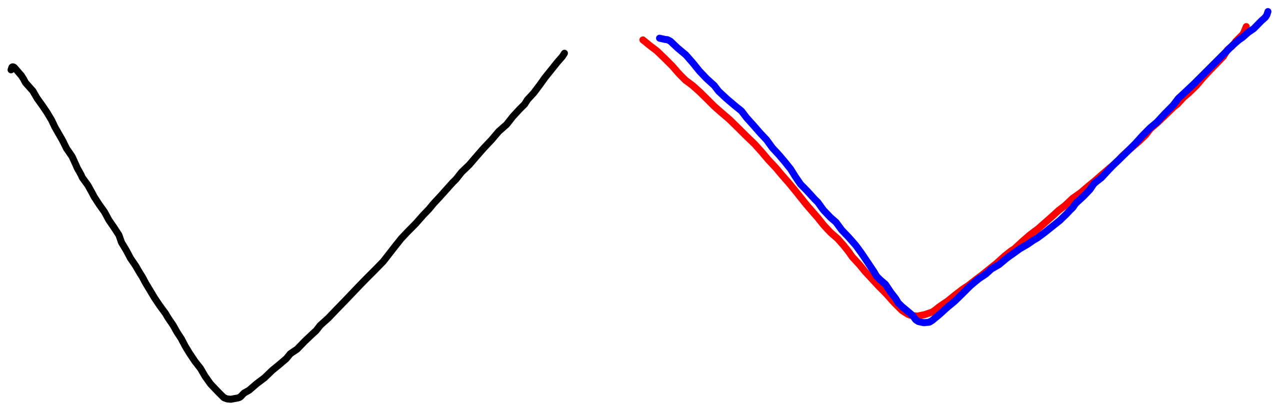




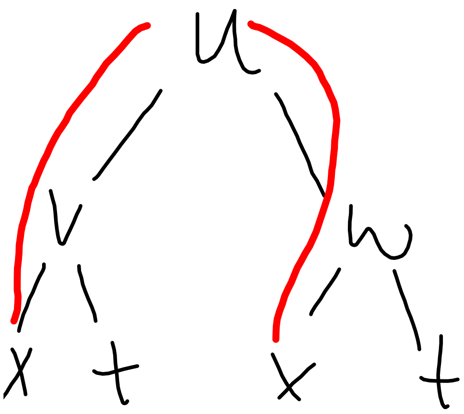


D'Alembert's solution

The strategy is to introduce new variables

$v = x + ct$ and

$w = x - ct$



Chain rule:

$$\cdot\ u_x = u_v \cdot \underbrace{v_x}_{=1} + u_w\, \underbrace{w_x}_{=1}$$

$$= u_v + u_w$$

$$\cdot\ u_{xx} = (u_v + u_w)_x$$

$$= (u_v + u_w)_v \cdot v_x + (u_v + u_w)_w\, w_x$$

$$= u_{vv} + 2u_{vw} + u_{ww}$$

Similarly,

$$u_{tt} = c^2(u_{vv} - 2u_{vw} + u_{ww})$$

Plug this into the wave equation:

$$u_{tt} = c^2 u_{xx}$$

$$\Rightarrow c^2(u_{vv} - 2u_{vw} + u_{ww})$$
$$= c^2(u_{vv} + 2u_{vw} + u_{ww})$$

$$\Rightarrow u_{vw} = 0$$

We have transformed the wave equation into a simpler equation:

$$\boxed{\frac{\partial^2 u}{\partial w \partial v} = 0}$$