Theorem: If U_1 and U_2 are solutions of a homogeneous linear PDE in some region R, then

$$u = C_1 U_1 + C_2 U_2$$

is also a solution of that PDE in region R, for $c_1, c_2 \in \mathbb{R}$.

Some of the simplest situations arise when we can solve the PDE as an ODE.

Example: $u(x, y)$

$$25 u_{yy} - 4u = 0$$

Since the ~~functio~~ PDE involves only derivatives with respect to y, we can treat it as an ODE.

$$25 u'' - 4u = 0$$

We know that solutions are of the form

$$u(y) = A e^{\lambda_1 y} + B e^{\lambda_2 y}$$

where λ_1, λ_2 are solutions to $25\lambda^2 - 4 = 0$

$$\lambda_1 = \frac{2}{5}, \quad \lambda_2 = -\frac{2}{5}$$

Solution: $u(x,y) = \underbrace{A(x)e^{2/5y} + B(x)e^{-2/5y}}_{\text{We have treated } x \text{ as a constant}}$

Example: Solve for $u = u(x,y)$, the equation

$$u_{xy} = u_x$$

Define $f(x,y) = u_x$ (change of function)

Then the equation becomes

$$f_y = f$$

The equation can be solved by writing it as

$$\frac{\partial f}{\partial y} = f \quad \Rightarrow \int \frac{\partial f}{f} = \int \partial y \quad \text{(separation)}$$

$$\Rightarrow \ln|f| = y + \tilde{c}(x)$$

$$\Rightarrow f = c(x)e^{y}$$

Recall $f = u_x = c(x)e^{y}$

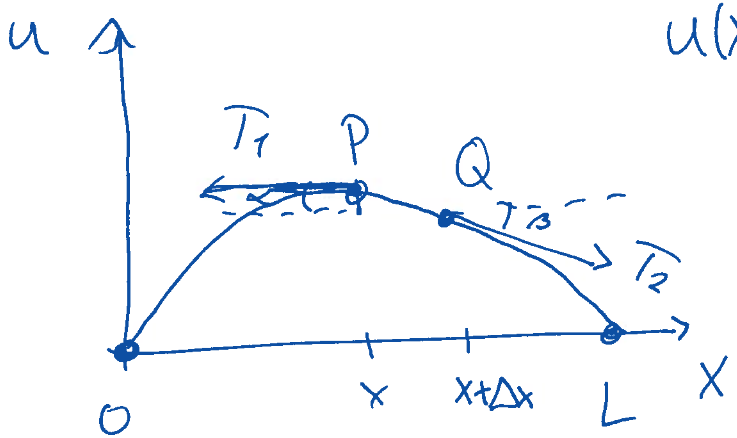
Now need to solve that PDE (ODE)

Integrating with respect to x:

$$u(x,y) = A(x)e^{y} + B(y)$$

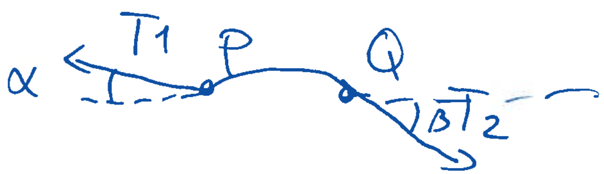
where A & B are arbitrary functions.

Modeling: Vibrating String, Wave equation (12.?)



$u(x,t)$: position (deflection) of the string at the point x & time $t>0$.

The string is on the x-axis, attached from 0 to L.
It is distorted and released.



Physical Assumptions

1. The mass of the string per unit of length is constant.
The string is perfectly elastic & does not offer any resistance to bending.

2. The initial tension in the string is large enough so that we can neglect the gravitational force.

3. The string performs small transverse vertical motions

Since the points move vertically, there is no motion in the horizontal direction, and thus the horizontal component is constant:

$$T_1 \cos\alpha - T_2 \cos\beta = 0$$

$$\Rightarrow \boxed{T_1 \cos\alpha = T_2 \cos\beta = T}$$

In the vertical direction: $\sum$ tensions $= ma$ (Newton's 2nd law)

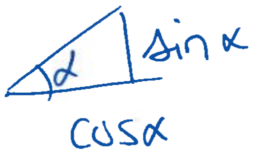
Let ρ be the mass density:

$$\boxed{T_2 \sin\beta - T_1 \sin\alpha = \rho \Delta x \frac{\partial^2 u}{\partial t^2}}$$

We divide both sides of the last equation by the first equation. $T_1 \cos\alpha = T_2 \cos\beta = T$

$$\frac{\cancel{T_2} \sin\beta}{\cancel{T_2}\cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{\rho}{T} \Delta x \frac{\partial^2 u}{\partial t^2}$$

$$\Longrightarrow \tan\beta - \tan\alpha = \frac{\rho}{T} \Delta x \frac{\partial^2 u}{\partial t^2}$$



$\rightsquigarrow \tan\alpha = \text{slope}$

Thus, $\tan\alpha = \left(\frac{\partial u}{\partial x}\right)\Big|_{x}$ ↑evaluated $\qquad \tan\beta = \left(\frac{\partial u}{\partial x}\right)\Big|_{x+\Delta x}$

$$\Longrightarrow \frac{1}{\Delta x}\left[\left(\frac{\partial u}{\partial x}\right)\Big|_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)\Big|_{x}\right] = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}$$

Taking limit $\Delta x \to 0$, we get

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}}$$

(Recall $\displaystyle\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x)$)

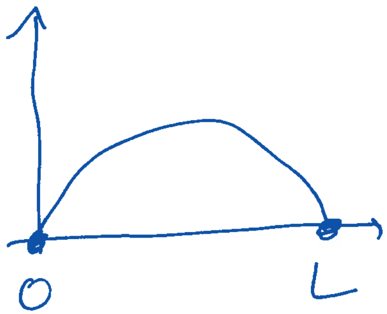
here $f = \left.\dfrac{\partial u}{\partial x}\right|_x$

<u>Wave equation</u>: $$\boxed{\frac{\partial^2 u}{\partial t^2} = \overbrace{c^2}^{T/\rho}\frac{\partial^2 u}{\partial x^2}}$$

Solution to the wave equation

$$R = \underbrace{[0, L]}_{x} \times \underbrace{[0, \infty)}_{t}$$

Boundary conditions: $u(0,t) = 0$, $u(L,t) = 0$ for all $t \geq 0$



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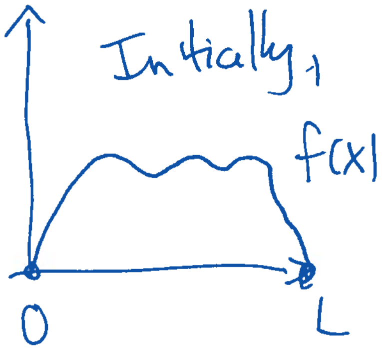
Initial conditions



$u(x, \overset{t=0}{0}) = f(x)$

↑ initial position (deflection) $x \in [0, L]$

~~and~~ $u_t(x,0) = g(x)$

↑ initial velocity

We will only look (at the beginning) for solutions of the form $u(x,t) = F(x)\,G(t)$

Step 1: Separation of variables: we find all solutions $u(x,t) = F(x)\,G(t)$

Get ODEs.

Step 2: Solve those ODEs using the boundary conditions.

Step 3: We sum all the solutions to get a solution that satisfies the initial conditions $\rightsquigarrow$ Fourier series.

Step 1 Determine solutions $u(x,t) = F(x)G(t)$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2 (F(x)G(t))}{\partial t^2} = c^2 \frac{\partial^2 (F(x)G(t))}{\partial x^2}$$

$$\Rightarrow F(x)\ddot{G}(t) = c^2 F''(x) G(t)$$

$$\Rightarrow \frac{\ddot{G}(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = k \leftarrow \text{constant.}$$

Since the functions are the same for all $x \in [0, L]$ and all $t \geqslant 0$, they need to be constant.

This gives two ODEs

1) $F'' - kF = 0$

2) $\ddot{G} - c^2 k G = 0$

Step 2: We use the boundary conditions:

$0 = u(0,t) = F(0)G(t)$ $\quad$ $0 = u(L,t) = F(L)G(t)$ for all $t \geq 0$

We exclude the case $G(t) = 0$, since then $u = 0$, and that gives a trivial solution.

$\Rightarrow$ $F(0) = 0$ & $F(L) = 0$