Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$\hookrightarrow$ 2π-periodic

Euler formula

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

These come from the <u>orthogonality relations</u>

1. $$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0$$
$n \neq m$

2. $$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0$$
$n \neq m$

3. $$\int_{-\pi}^{\pi} \cos nx \sin mx \, dx = 0$$

To compute these integrals, use trigonometric identities

$$\cos nx \cos mx = \frac{1}{2}\left[\cos(n+m)x + \cos(n-m)x\right]$$

$$\sin nx \sin mx = \frac{1}{2}\left[\cos(n-m)x - \cos(n+m)x\right]$$

$$\sin nx \cos mx = \frac{1}{2}\left[\sin(n+m)x + \sin(n-m)x\right]$$

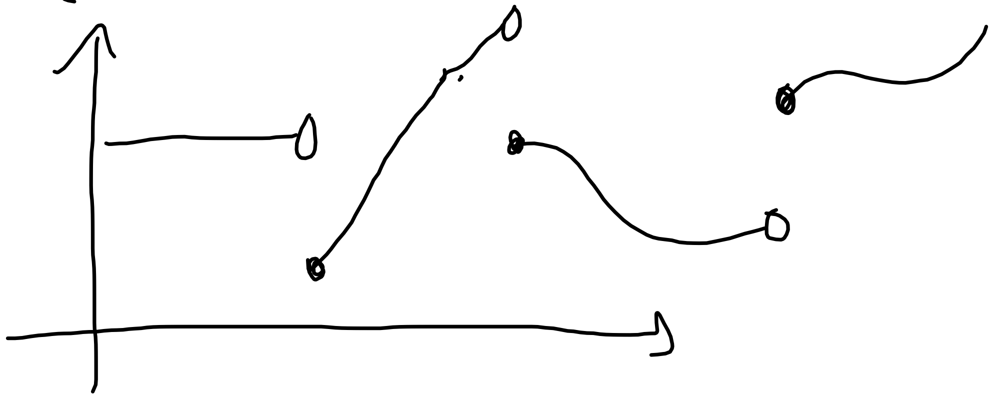
When are 2π-periodic functions represented as Fourier series?

<u>Theorem</u>: Let $f(x)$ be piecewise continuous. Assume also that it has left and right derivatives. Then the Fourier series with coefficients given by Euler formula converges to $f(x)$.

At the points x_0 where $f(x)$ is not continuous, the series converges to

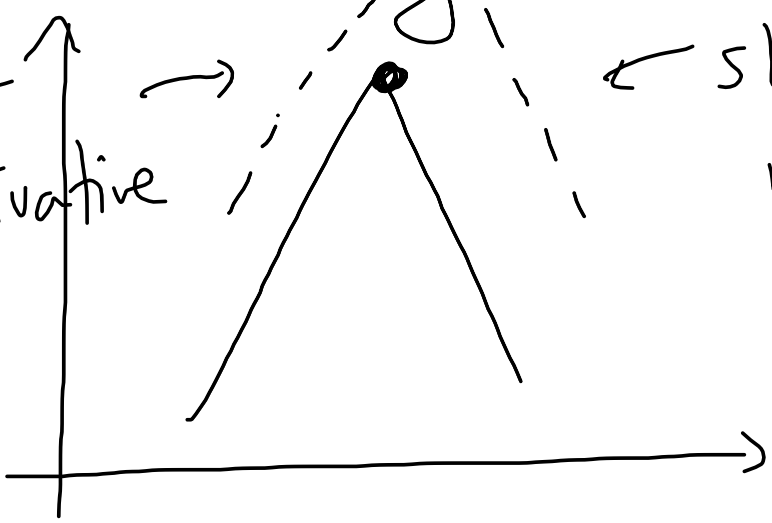
$$\frac{1}{2}\left(\lim_{x \to x_0^-} f(x) + \lim_{x \to x_0^+} f(x)\right)$$

- Piecewise continuous

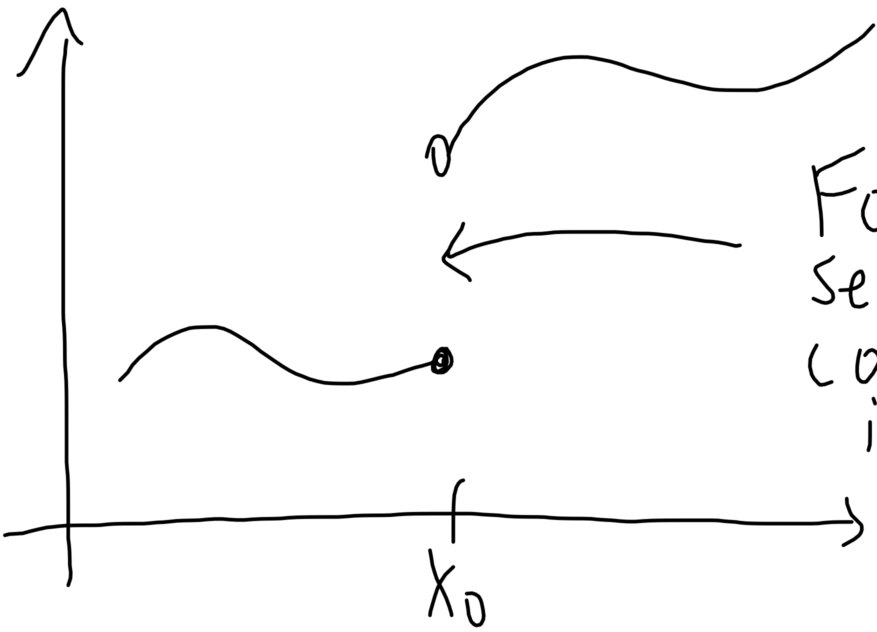


- left & right derivatives

left derivative



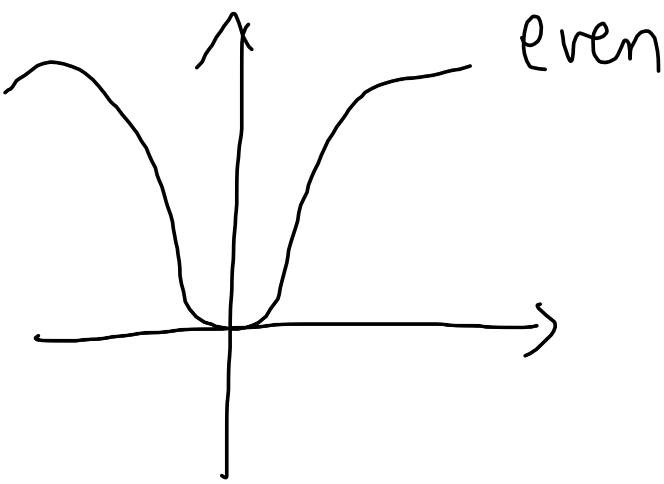
slope = right deriv.

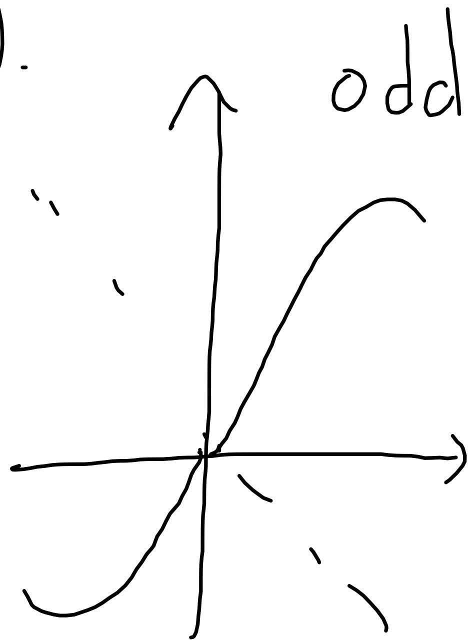
- 

Fourier series converges in the middle at that point

Recall

Definition: A function $f(x)$ is called even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$.





Ex: · x^n is even if n is even

is odd if n is odd

· $\cos x$ is even

· $\sin x$ is odd

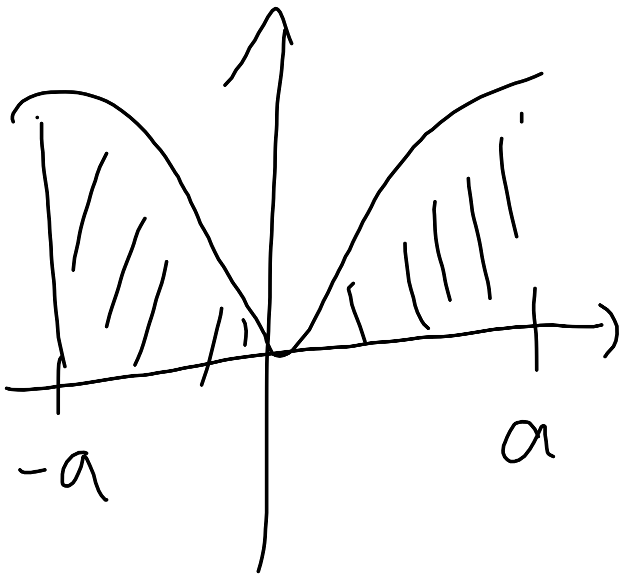
Theorem

If $f(x)$ is even

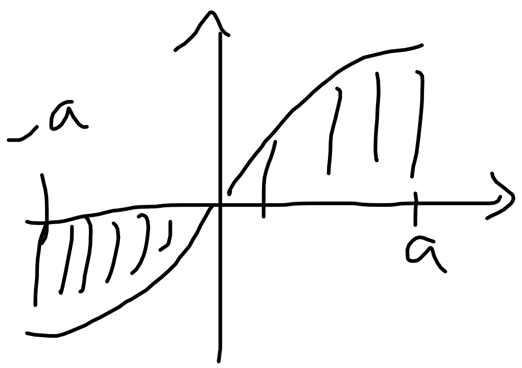
$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

If $f(x)$ is odd

$$\int_{-a}^{a} f(x)\,dx = 0$$



$= 2 \times$ Area $[0, a]$



$= 0$

If $f(x)$ & $g(x)$ are both even or are both odd

$\Rightarrow f(x)g(x)$ is even

If $f(x)$ & $g(x)$ have different "parity"

$\Rightarrow f(x)g(x)$ is odd

One is even & the other one is odd

In particular,

1) if $f(x)$ is even then

$f(x)\cos nx$ is even

$f(x)\sin nx$ is odd

2) if $f(x)$ is odd

$f(x)\cos nx$ is odd

$f(x)\sin nx$ is even

$$\Rightarrow 1) \ \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx =$$

$f(x)$ even

$$\frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx \, dx$$

$$= a_n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx$$

$$= 0$$

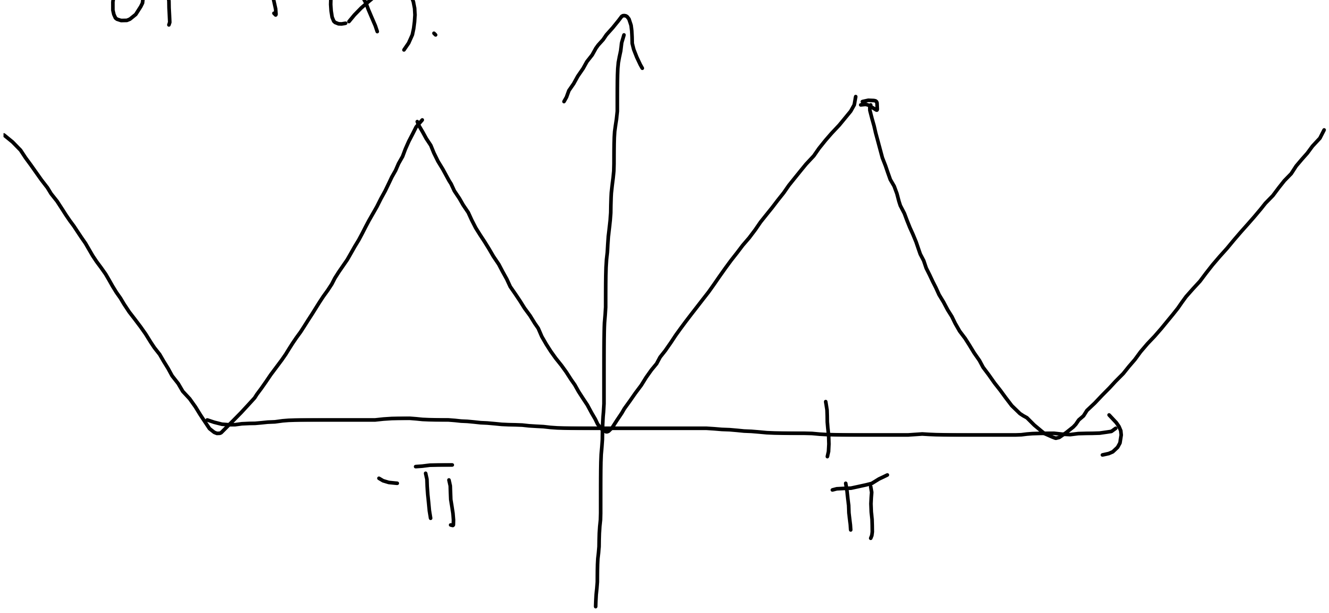
$$2) \ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

f(x) is odd

Example: Let $f(x) = |x|$ for $-\pi < x < \pi$ and assume $f(x)$ is 2π-periodic. Find the Fourier series of $f(x)$.



We need to find the Euler coefficients a_0, a_n, b_n.

The function is <u>even</u> so $b_n = 0$ for all n.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \, dx$$

$$= \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \cos nx \, dx$$

Integration by parts

$$\int u\,dv = uv - \int v\,du$$

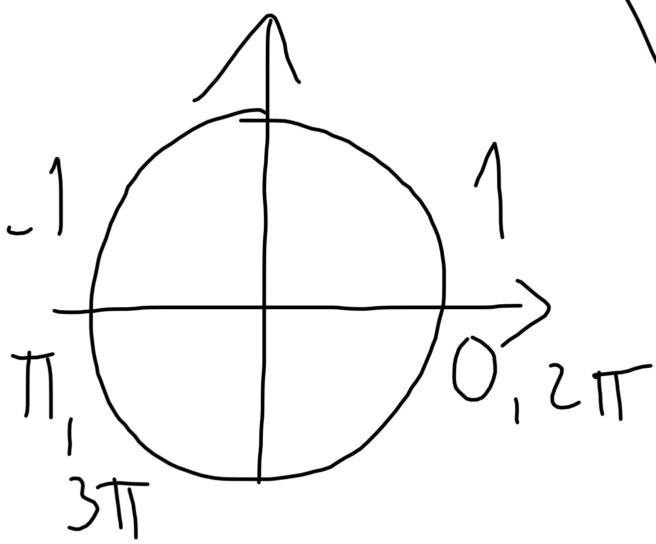
Choose $u = x$ $\quad dv = \cos nx\,dx$

$du = dx \quad v = \frac{1}{n}\sin nx$

$$= \frac{2}{\pi}\left[\frac{x}{n}\sin nx - \frac{1}{n}\int \sin nx\,dx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\cancel{\frac{x}{n}\sin nx} + \frac{1}{n^2}\cos nx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\cos 0\right]$$



$$\cos n\pi = \begin{cases} 1 \text{ if } n \text{ is even} \\ -1 \text{ if } n \text{ is odd} \end{cases}$$

$$= (-1)^n$$

$$= \frac{2}{\pi}\left[\frac{1}{n^2}(-1)^n - \frac{1}{n^2} \right]$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{-4}{n^2 \cdot \pi} & \text{if } n \text{ is odd} \end{cases}$$

$$= a_n$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \; \cancel{+ \underbrace{b_n}_{=0} \sin nx}$$

$$= \frac{\pi}{2} + \sum_{n \text{ odd}} \frac{-4}{n^2 \pi} \cos nx$$

Change of index

$$n \rightsquigarrow 2n+1$$

$$= \frac{\pi}{2} + \sum_{n=0}^{\infty} \frac{-4}{(2n+1)^2 \pi} \cos((2n+1)x)$$

$$\frac{\pi}{2} + \frac{-4}{\pi(1^2)}\cos x$$

$$+ \frac{-4}{\pi(3^2)}\cos 3x$$

$$+ \frac{-4}{\pi(5^2)}\cos 5x + \dots$$

$$\frac{\pi}{2} + \frac{-4}{\pi(1^2)}\cos x$$

$$+ \frac{-4}{\pi(3^2)}\cos 3x + \dots$$

$$\boxed{f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}\cos(2n+1)x}$$

$$\begin{array}{c} \| \\ |x| \end{array}$$

Use this series to compute $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$

Choose $x = 0$

$$f(0) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}\underbrace{\cos 0}_{=1}$$

$$\Rightarrow |0| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

$$\Rightarrow \boxed{\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}}$$

<u>Example</u>: Let $f(x) = x$ for $-\pi \le x < \pi$ be 2π-periodic. Find the Fourier series of f.

Since $f(x)$ is odd,
$a_n = 0$ for $n = 0, 1, 2, \ldots$
We only need to compute b_n.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \sin nx \, dx$$

We compute the integral using integration by parts.

$$u = x \qquad dv = \sin nx \, dx$$

$$du = dx \qquad v = -\frac{1}{n}\cos nx$$

$$= \frac{2}{\pi}\left[-\frac{x}{n}\cos nx + \frac{1}{n}\int \cos nx \, dx \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{x}{n}\cos nx + \cancel{\frac{1}{n^2}\sin nx} \right]_0^{\pi}$$

$$= \frac{2}{\cancel{\pi}}\left[-\frac{\cancel{\pi}}{n}(-1)^n - 0 \right]$$

$$= 2\frac{(-1)^{n+1}}{n}$$

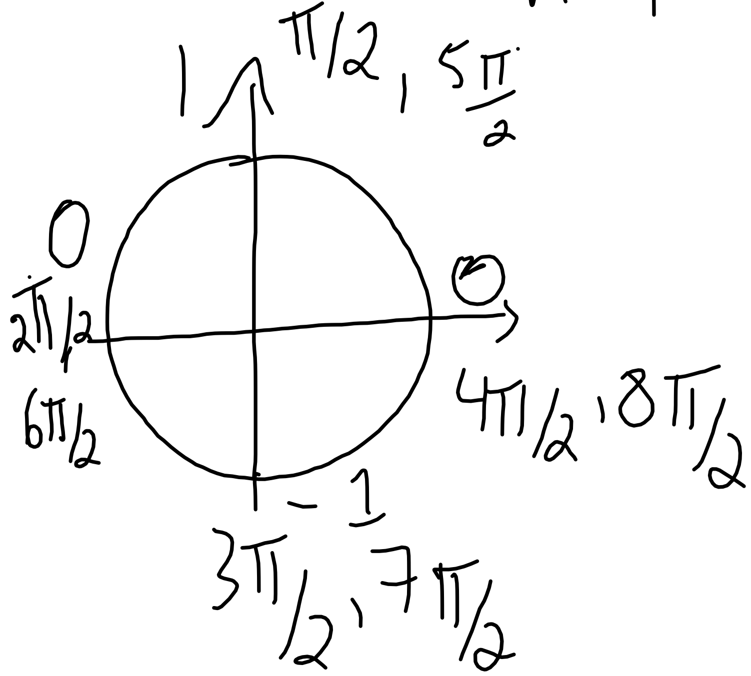
$$\boxed{\text{Fourier series} \quad \underset{\overset{\parallel}{x}}{f(x)} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx}$$

Use this Fourier series to find the value of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

Choose $x = \frac{\pi}{2}$.

$$\frac{\pi}{2} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin\left(n\cdot\frac{\pi}{2}\right)$$



$$= \begin{cases} 0 & \text{if } n \text{ even} \\ 1 & \text{if } n = 1, 5, 9, \ldots \\ -1 & \text{if } n = 3, 7, 11, \ldots \end{cases}$$

$$\to = 2 \sum_{n \text{ odd}} \frac{(-1)^{n+1}}{n} \cdot \sin n\pi/2$$

Change of index

$$n \rightsquigarrow 2n-1$$

$$= 2 \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{2n-1} \underbrace{\left(\sin(2n-1)\cdot \pi/2\right)}_{(-1)^{n+1}}$$

$$= 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}}$$