Overview

Fourier analysis &
Laplace transforms

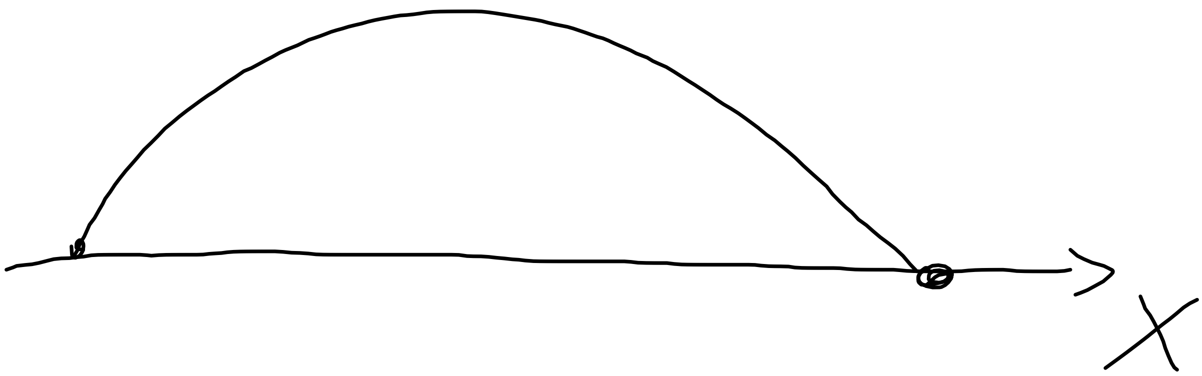
Example: Original motivation was to solve the heat equation on a metal plate
$u(x,y,t)$ temperature in the plate at time t,

position (x, t), satisfies

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

<u>Example</u>: Vibrating String

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$



$u(x,t)$ position at point x & time t

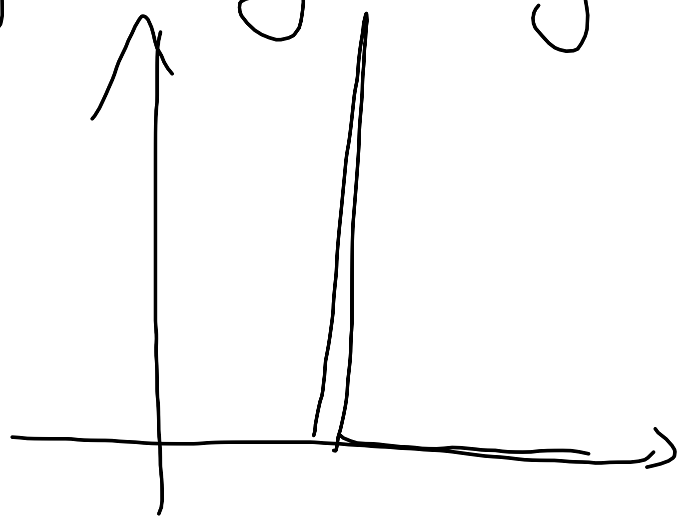
Fourier Series: sum of sinusoidal waves

$$f(x) = \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Laplace transforms

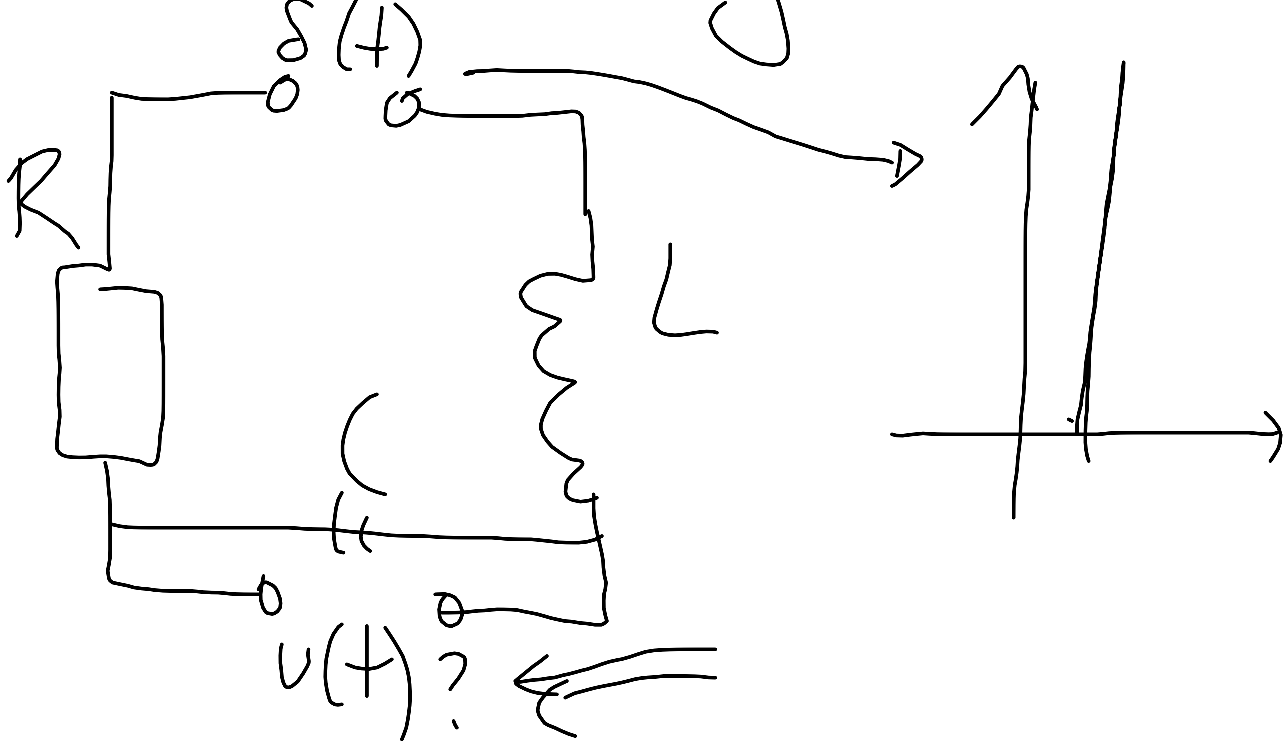
Help solve ODEs
(ordinary differential equations)

$$y'' + ay' + by = r(t)$$



$r(t)$

<u>Example</u>: Consider the following circuit



R resistor C capacitor
L inductor

Kirchoff's voltage law

$$Lq'' + Rq' + \frac{1}{C}q = v(t)$$

q charge

Laplace transform

$$f \longrightarrow \mathcal{L}(f)$$

function $\longmapsto$ function

$$\begin{tikzcd} \text{Diff. eq} \arrow[r, bend left, "\mathcal{L}"] & \text{alg. Equation} \arrow[l, bend left, "\mathcal{L}^{-1}"] \end{tikzcd}$$

(Much easier to solve)

Numerical analysis

Exact methods rarely work, instead need

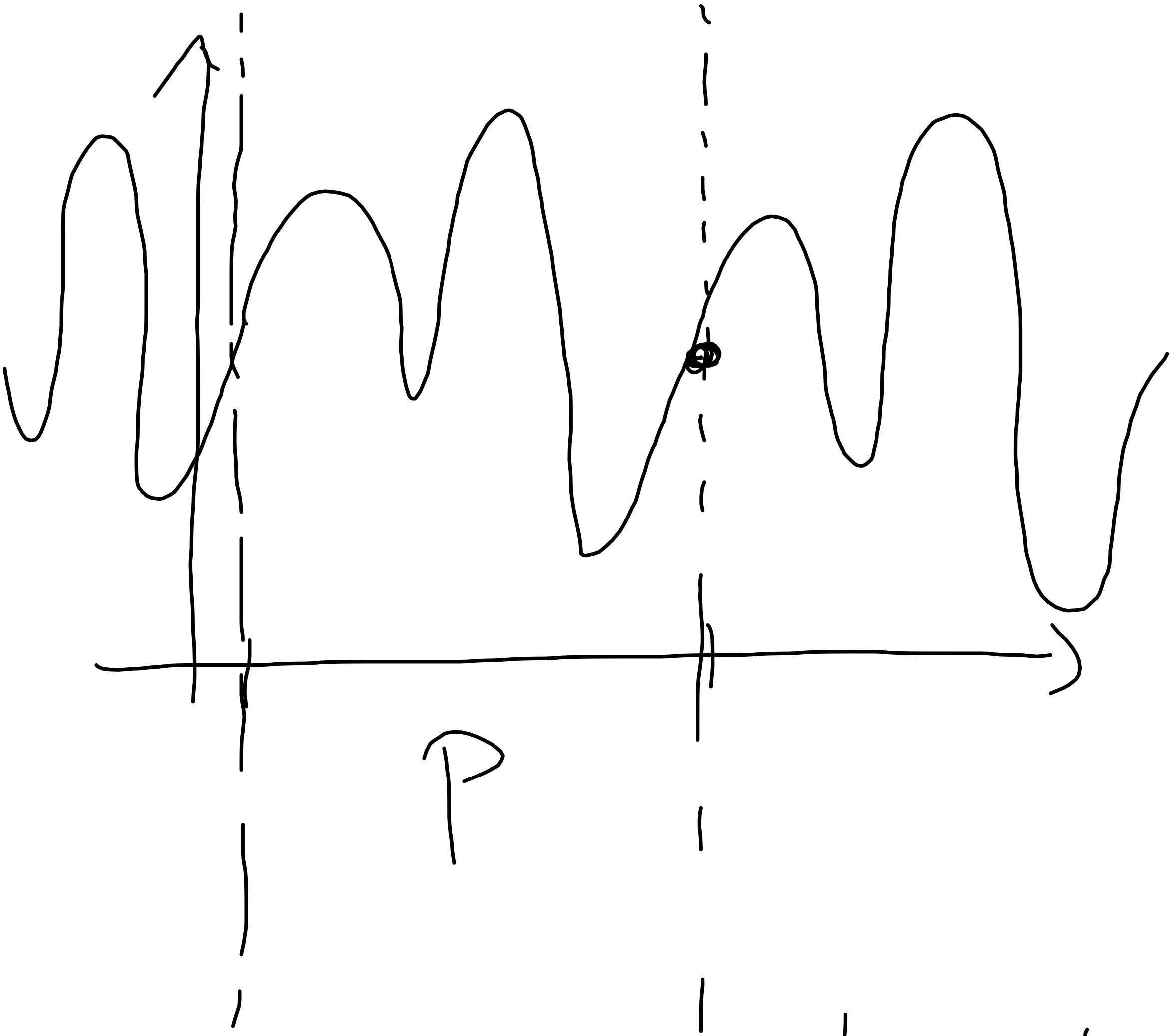
approximation methods.

- $F(x) = 0$
- $\int_a^b f(x)\,dx$
- ODEs and PDEs (partial differential equations)

Periodic Functions

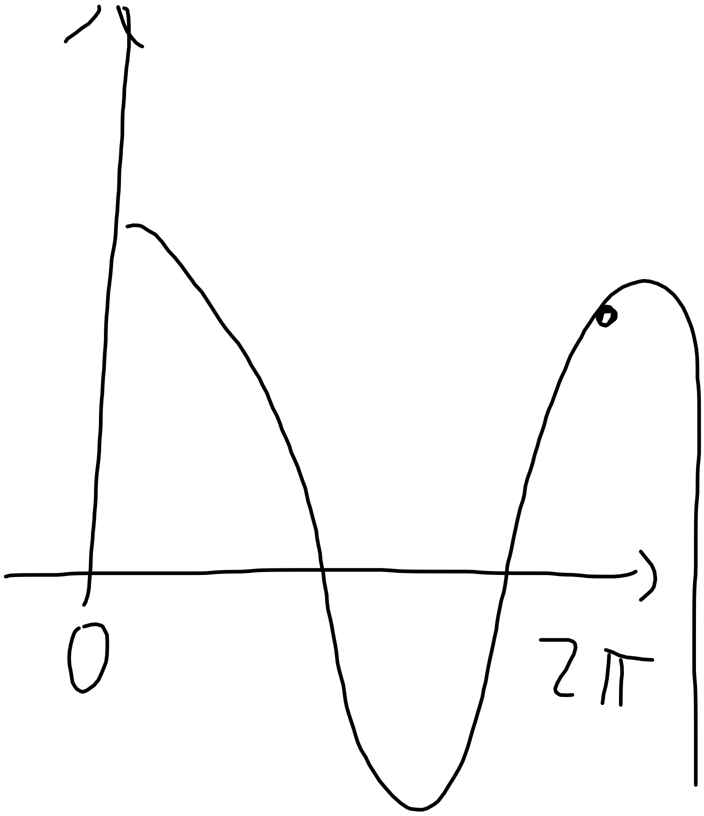
Def: A function $f: \mathbb{R} \to \mathbb{R}$ is called *periodic* if there is some positive number p, called a *period*, such that

$$f(x+p) = f(x) \text{ for all } x \in \mathbb{R}.$$

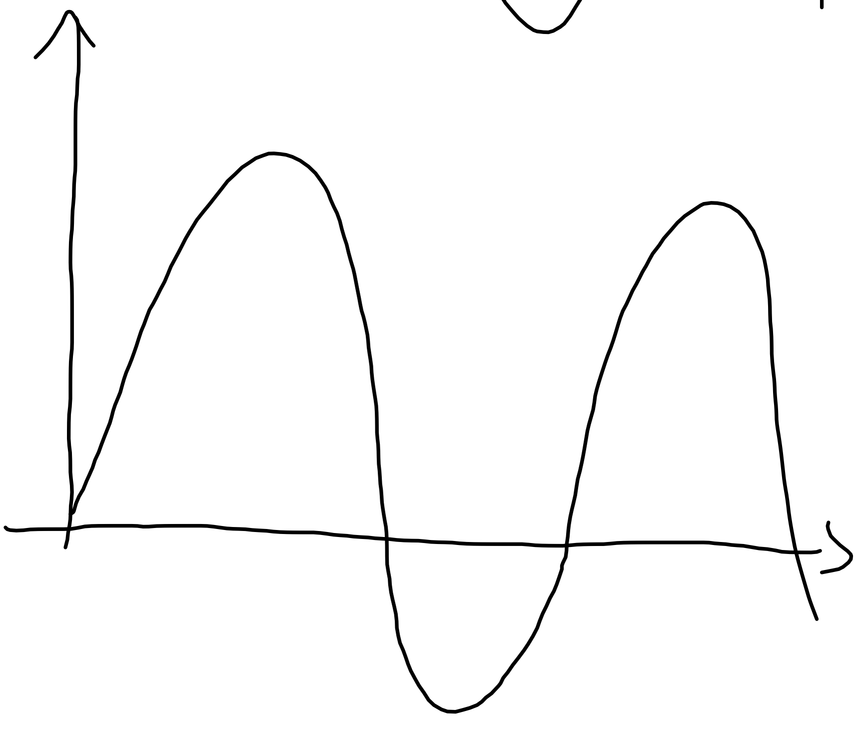


It is a function that repeats a pattern of length P.

$\cos(x)$
periodic
$(p = 2\pi)$



$\sin(x)$
$(p = 2\pi)$



Properties:

$f(x)$ and $g(x)$ of period p.

1. $f(x) + g(x)$ has period p.

2. $\lambda \in \mathbb{R} \setminus \{0\}$

   $\lambda \cdot f(x)$ has period p.

3. $f(x)g(x)$ has period p.

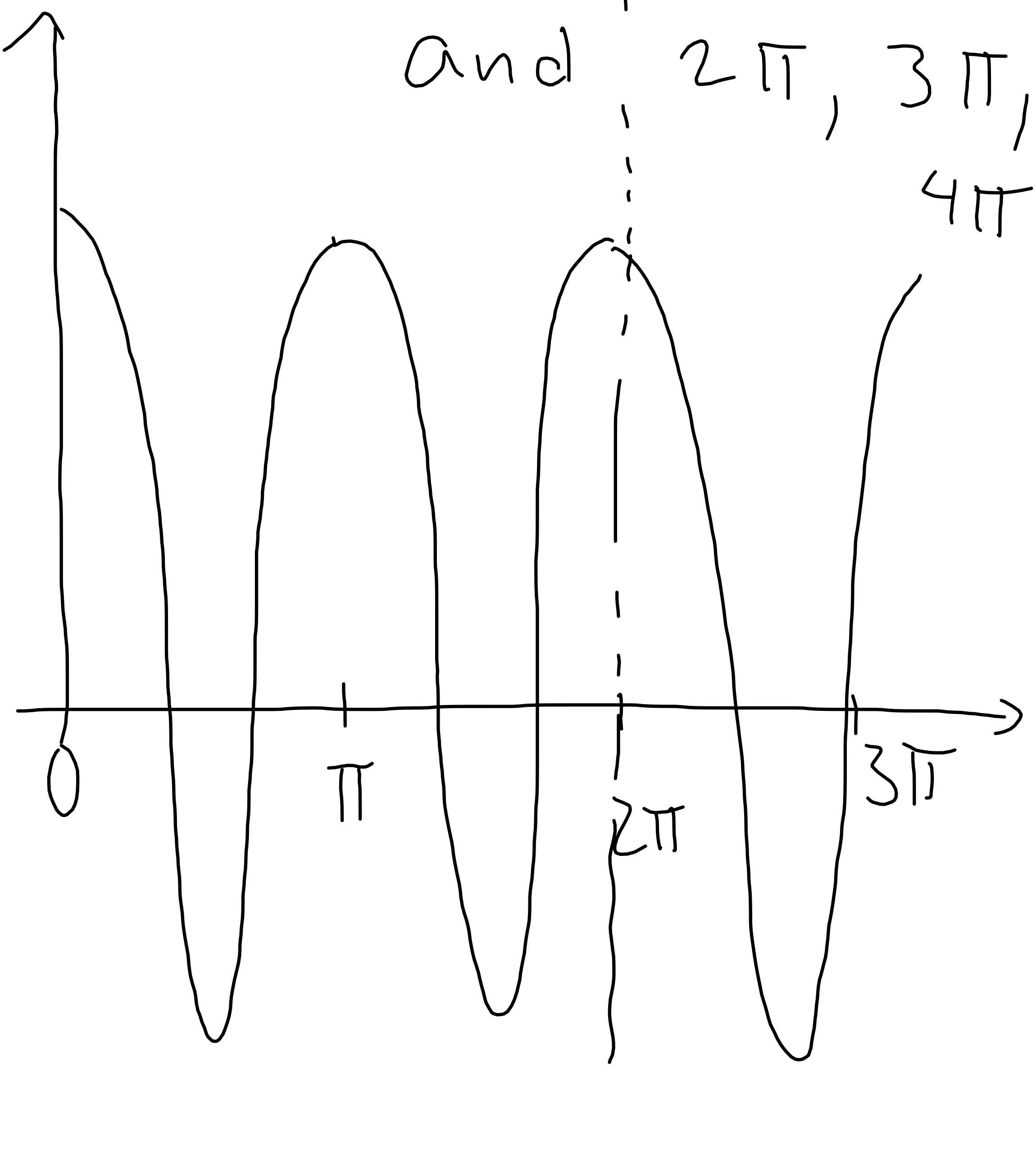
4. $\dfrac{f(x)}{g(x)}$ has period p.

Important: The period is not unique.

If $f(x)$ has period p, then np is also a period for any integer n.

cos(2x) has period π

and 2π, 3π, 4π,



We call the smallest period of a function, the <u>fundamental period</u>.

Ex.: $\cos(2x)$ has fundamental period π.

<u>Goal</u>: Represent functions $f(x)$ of period 2π in terms of trigonometric

system:

$1, \cos x, \cos(2x), \cos(3x), \ldots$

$\sin x, \sin(2x), \sin(3x), \ldots$

<u>Fourier Series</u>

We consider <u>trigonometric series</u>:

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \ldots$$

By "the above properties", if the series converges, then its sum is of period 2π.

<u>Definition</u>: A <u>Fourier Series</u> is a trigonometric series that converges to a function $f(x)$.

$\hookrightarrow$ (necessarily of period 2π)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

↳ Fourier series of f

f is <u>represented</u> by this series.

Given $f(x)$, how do we find the coefficents $a_0, a_1, \ldots$ $b_1, \ldots$ of the Fourier series of f?

Euler Formula

If $f(x)$ is represented by a Fourier series, then its cofficients are given by

1. $a_0 = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$ → mean value of $f(x)$

2. $a_n = \dfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ $n=1,2,3,\ldots$

3. $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$n = 1, 2, \ldots$

We can approximate $f(x)$ by its <u>Fourier Polynomials</u>

$$f(x) \approx a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$$

To prove Euler Formula, one needs the <u>orthogonality relations</u>

1. $$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0 \quad (n \neq m)$$

2. $$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0 \quad (n \neq m)$$

3. $$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0$$

For all n, m.

Let's prove Euler Formula for a_n:

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

(want $= a_n$)

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \left(a_0 + \sum_{k=1}^{\infty} \left(a_k \cos kx + b_k \sin kx \right) \right) \cdot \cos nx \, dx$$

$$= \frac{1}{\pi} \Bigg[\cancelto{0}{\int_{-\pi}^{\pi} a_0 \cos nx \, dx}$$

use orthogonality

$$+ \sum_{k=1}^{\infty} \int_{-\pi}^{\pi} a_k \cos kx \cdot \cos nx \, dx$$

$$+ \sum_{k=1}^{\infty} \int_{-\pi}^{\pi} b_k \sin kx \cos nx \, dx \Bigg]$$

use ortho

These are all $= 0$,
except in the second
one when $k = n$.

$$= \frac{1}{\pi} a_n \int_{-\pi}^{\pi} \cos nx \cdot \cos nx \, dx$$

trigo. identity

$$= \frac{1}{\pi} a_n \int_{-\pi}^{\pi} \frac{1 + \cos 2nx}{2} \, dx$$

$$= a_n.$$