Fourier series

f $2L$-periodic

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right)$$

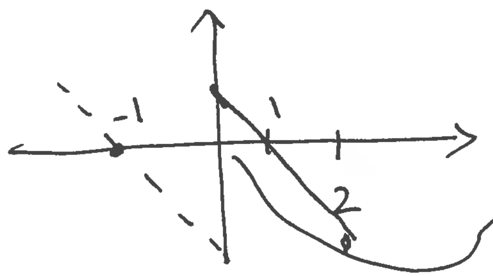
$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

<u>Example</u> (Fall 2015 #1)

Let $f(x) = 1 - x$ ~~be 2-periodic~~, defined $[0,2]$

Consider the odd extension of period 4, $g(x)$.

Find the Fourier series of $g(x)$.



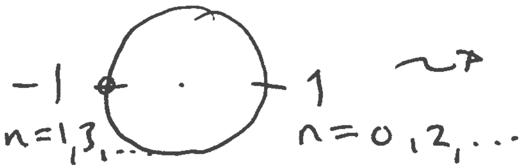
$g(-x) = -g(x)$

→ Fourier series converges at the average.

~~Since~~ We compute the Fourier sine series of $f(x)$
(since $g(x)$ is the odd extension)

$$\underset{L \rightsquigarrow 2}{b_n} = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi}{2}x\right)dx = \int_0^2 (1-x)\sin\left(\frac{n\pi}{2}x\right)dx$$

$$= \frac{2}{n\pi}\left(1+\cos n\pi\right)$$



$$\rightsquigarrow \cos n\pi = \begin{cases} 1 & \text{if } n \text{ even} \\ -1 & \text{if } n \text{ odd} \end{cases} = (-1)^n$$

$$= \frac{2}{n\pi}\left(1+(-1)^n\right) = \begin{cases} 0 & \text{if } n \text{ odd} \\ \frac{4}{n\pi} & \text{if } n \text{ even} \end{cases}$$

$\rightsquigarrow$ Replace $n \rightsquigarrow 2m$

Fourier series: $\displaystyle\sum_{m=1}^{\infty} \frac{\cancel{4}\,2}{\cancel{2}m\pi}\sin\left(\frac{\cancel{2}m\pi}{\cancel{2}}x\right)$

Solving PDEs with Fourier series

Wave equation: $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

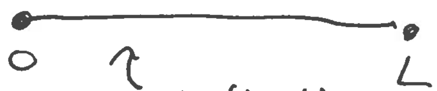
Boundary conditions: $u(0,t) = 0 = u(L,t)$ for all $t \geq 0$

Initial conditions: $u(x,0) = f(x)$ initial position

$u_t(x,0) = g(x)$ initial velocity

$0 \leq x \leq L$

Heat equation: $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$



$u(x,t)$ = temperature at x, time t

Boundary conditions (varies)

Initial condition: $u(x,0) = f(x)$ initial temperature

Example (Fall 2015 #6a): PDE $\dfrac{\partial^2 u}{\partial t^2} - 2\dfrac{\partial u}{\partial t} + u = \dfrac{\partial^2 u}{\partial x^2}$

$0 < x < \pi, \ t \geq 0$

Find all solutions of the form $u(x,t) = F(x)G(t)$ that satisfy the boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(\pi, t) = 0 \quad t \geq 0$$

1) Suppose $u(x,t) = F(x)G(t)$. Plug this into the PDE to get two ODEs.

$$FG'' - 2FG' + FG = F''G$$

$$\Rightarrow F(G'' - 2G' + G) = F''G$$

$$\Rightarrow \frac{F''}{F} = \frac{G'' - 2G' + G}{G} = K \text{ constant}$$

Two ODEs: $\begin{cases} F'' - kF = 0 \\ G'' - 2G' + G = KG \end{cases}$

2) Use the boundary conditions: $F'(0)G(t) = 0 \Rightarrow F'(0) = 0$

$$F'(\pi) = 0$$

Three cases: $\underbrace{k = p^2}_{\text{trivial}}$, $\underbrace{k = 0}_{F(x) = A}$, $k = -p^2$

$\underline{k = 0}$: 1st ODE $F'' = 0 \Rightarrow F(x) = A + Bx$

$$\left.\begin{aligned} &\Rightarrow F'(0) = 0 \Rightarrow B = 0 \\ &F'(\pi) = 0 \Rightarrow B = 0 \end{aligned}\right\} F(x) = A \text{ is a solution.}$$

2nd ODE: $G'' - 2G' + G = 0$

$$\rightsquigarrow G(t) = Ce^t + Dte^t$$

$\underline{k = -p^2}$: 1st ODE: $F(x) = A\cos px + B\sin px$, $F'(x) = -Ap\sin px + Bp\cos px$

$$F'(0) = 0 \Rightarrow Bp = 0 \Rightarrow B = 0$$

$$F'(\pi) = 0 \Rightarrow -Ap\sin p\pi = 0$$

$$\Rightarrow \underbrace{A = 0}_{\text{trivial}} \text{ or } \sin p\pi = 0$$

$$\Rightarrow p = 1, 2, 3, \ldots$$

$F(x) = A\cos px$, $p = 1, 2, 3, \ldots$

2^{nd} ODE: $G'' - 2G' + G = -p^2 G$

$$\rightsquigarrow G(t) = e^t (C \cos pt + D \sin pt)$$

$$\Rightarrow u_p(x,t) = e^t (C \cos pt + D \sin pt) \cos px \quad p = 1, 2, 3, \ldots$$

The End

Normally: 3) If initial conditions, sum over all $\sum_{p=1}^{\infty} u_p(x,t)$ and use the initial conditions to find values for the coefficients via Fourier series.

Laplace transforms

f defined $t \geq 0$

$$F(s) = \mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)\, dt$$

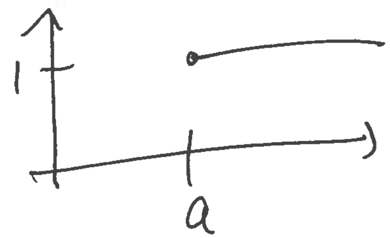
Solving ODEs:

$$t\text{-world} \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} s\text{-world}$$

$$y'' + ay' + by = r(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} \mathcal{L}(y) = \frac{(s+a)y(0) + y'(0)}{s^2+as+b} + \frac{\mathcal{L}(r)}{s^2+as+b}$$

Unit step function

$$u(t-a) = \begin{cases} 0 & \text{if } t \leq a \\ 1 & \text{if } t > a \end{cases}$$



Fonctions by parts:

$$f(t) = \begin{cases} g(t) & \text{if } 0 < t < a \\ h(t) & \text{if } t \geq a \end{cases}$$

$$= g(t)\big(u(t-0) - u(t-a)\big) + h(t)u(t-a)$$

Example: (August 2016, #1)

Solve $y'' + 3y' + 2y = t\,u(t-1)$, $y(0)=1$, $y'(0)=-1$

Apply $\mathcal{L}$ on both sides:

$$\left(s^2\mathcal{L}(y) - s\,(y(0)) - y'(0)\right) + 3\left(s\mathcal{L}(y) - y(0)\right) + 2\mathcal{L}(y)$$

$$= e^{-s}\mathcal{L}(t+1) = e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) = e^{-s}\left(\frac{s+1}{s^2}\right)$$

$$\Rightarrow \mathcal{L}(y) = \frac{e^{-s}\frac{s+1}{s^2} + (s+2)}{s^2+3s+2} = \frac{e^{-s}\frac{s+1}{s^2} + (s+2)}{(s+2)(s+1)}$$

$$= \frac{1}{s+1} + e^{-s}\cdot\frac{1}{s^2(s+2)}$$

$$\rightsquigarrow \frac{1}{s^2(s+2)} = \frac{A}{s+2} + \frac{B}{s} + \frac{C}{s^2} \Rightarrow \begin{array}{l} C = 1/2 \\ B = -1/4 \\ A = 1/4 \end{array}$$

$$\mathcal{L}(y) = \frac{1}{s+1} + e^{-s}\left(\frac{1}{4}\cdot\frac{1}{s+2} - \frac{1}{4}\cdot\frac{1}{s} + \frac{1}{2}\frac{1}{s^2}\right)$$

Apply inverse LT: $y = e^{-t} + \left(\frac{1}{4}\cdot e^{-2(t-1)} - \frac{1}{4} + \frac{1}{2}(t-1)\right) u(t-1)$

Fourier transforms

Example (August 2016, #8)

$$\frac{\partial u}{\partial t} = t\frac{\partial^2 u}{\partial x^2} \qquad x \in \mathbb{R},\ t \geq 0 \text{ with initial conditions}$$

$$u(x,0) = e^{-x^2/2}.$$

We apply FT with respect to x.

$$\mathcal{F}(u_t)^{(\omega,t)} = \mathcal{F}(\underset{\text{constant}}{\underbrace{t}}\, u_{xx})^{(\omega,t)}$$

$$\| $$

$$\frac{\partial}{\partial t}\mathcal{F}(u)^{(\omega,t)} = t\,\mathcal{F}(u_{xx}) = \cancel{t} -\omega^2 t\,\mathcal{F}(u)^{(\omega,t)}$$

↳ formula for derivatives

This is an ODE of t

Can solve by separation of variables

$$\int \frac{\frac{\partial \mathcal{F}(u)}{\partial t}}{\mathcal{F}(u)} = \int -\omega^2 t \, \partial t \Rightarrow \mathcal{F}(u) = C(\omega)\, e^{-\frac{\omega^2 t^2}{2}}$$

↳ Apply the inverse FT

From the initial conditions:

$$C(\omega) = \mathcal{F}(u)(\omega, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0)\, e^{-i\omega x}\, dx = \mathcal{F}(\underbrace{u(x,0)}_{\text{initial conditions}})$$

$$= \mathcal{F}(e^{-x^2/2})$$

$$= e^{-\omega^2/2}$$

Get u by inverse FT:

$$u = \sqrt{\frac{1}{t^2+1}}\, e^{-\frac{1}{2t^2+2} x^2}$$