Problem 1

Let f be the 2π-periodic functions defined by $f(x) = \cos\left(\frac{x}{2}\right)$ when $x \in [-\pi, \pi]$. Make a drawing of the function f for the interval $[-3\pi, 3\pi]$, and compute the Fourier series of f. Use the result to compute the value of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1-4n^2}.$$

Possible Solution

First observe that $f(-x) = \cos\left(\frac{-x}{2}\right) = \cos\left(\frac{x}{2}\right)$, so $f(x)$ is an even function. Then the Fourier series of the function $f(x)$ is of the form

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nx).$$

Then

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} \cos\left(\frac{x}{2}\right)\,dx = \frac{1}{\pi}\frac{\sin\left(\frac{x}{2}\right)}{\frac{1}{2}}\Bigg]_{x=0}^{x=\pi} = \frac{2}{\pi}, \end{aligned}$$

and

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} \cos\left(\frac{x}{2}\right)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left(\cos\left(\frac{2n+1}{2}x\right) + \cos\left(\frac{2n-1}{2}x\right)\right)\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} \cos\left(\frac{2n+1}{2}x\right) + \cos\left(\frac{2n-1}{2}x\right)\,dx \\ &= \frac{1}{\pi}\left(\frac{2}{2n+1}\sin\left(\frac{2n+1}{2}x\right) + \frac{2}{2n-1}\sin\left(\frac{2n-1}{2}x\right)\right]_{x=0}^{x=\pi} \\ &= \frac{1}{\pi}\left(\frac{2}{2n+1}\sin\left(\frac{2n+1}{2}\pi\right) + \frac{2}{2n-1}\sin\left(\frac{2n-1}{2}\pi\right) - 0 - 0\right) \\ &= \frac{1}{\pi}\left(\frac{2}{2n+1}(-1)^n + \frac{2}{2n-1}(-1)^{n+1}\right) = \frac{(-1)^n}{\pi}\left(\frac{2}{2n+1} - \frac{2}{2n-1}\right) \\ &= \frac{(-1)^n}{\pi}\left(\frac{-4}{4n^2-1}\right) = \frac{(-1)^n}{\pi}\left(\frac{4}{1-4n^2}\right). \end{aligned}$$

Thus, the Fourier series to $f(x)$ is

$$\frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{(-1)^n}{\pi}\frac{4}{1-4n^2}\cos(nx).$$

Finally to find the value of the series, we evaluate the Fourier series of $f(x)$ at $x = 0$, and by the Fourier Theorem and since $f(x)$ is continuous at $x = 0$ we have that

$$1 = \cos\left(\frac{0}{2}\right) = f(0) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{(-1)^n}{\pi} \frac{4}{1-4n^2} \cos(0) = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{1-4n^2} \,,$$

whence

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1-4n^2} = \frac{\pi}{4}\left(1 - \frac{2}{\pi}\right) = \frac{\pi - 2}{4} \,.$$

Problem 2

Find all the non-trivial solutions of the heat equation

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \qquad \text{where} \qquad 0 \le x \le 2\pi \qquad \text{and} \qquad t \ge 0 \,,$$

that are of the form $u(x,t) = F(x) \cdot G(t)$, and that satisfy the boundary conditions

$$u(0,t) = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial x}(2\pi, t) = 0 \qquad \text{for every } t \ge 0 \,.$$

Use this to find a solution satisfying the initial condition

$$u(x,0) = \cos(x)\sin\left(\frac{x}{4}\right) \qquad \text{for every } 0 \le x \le 2\pi \,.$$

Possible Solution

If $u(x,t) = F(x)G(t)$ we have that

$$u_{xx} = F''G \qquad \text{and} \qquad u_t = FG' \,.$$

Then if we replace this into the heat equation we have that $FG' = 2F''G$, so it follows that

$$\frac{F''}{F} = \frac{G'}{2G} = K \qquad \text{where } K \in \mathbb{R} \,.$$

From this we deduce the two following ODEs

$$F'' - KF = 0 \qquad \text{and} \qquad G' - 2KG = 0 \,.$$

Additionally, observe that from the boundary conditions it follows that

$$u(0,t) = F(0)G(t) = 0 \qquad \text{implies} \qquad F(0) = 0 \,,$$

$$u_x(2\pi, t) = F'(2\pi)G(t) = 0 \qquad \text{implies} \qquad F'(2\pi) = 0\,.$$

First we solve the ODE $F'' - KF = 0$.

Suppose that $\underline{K = 0}$, then $F'' = 0$, from where we have that $F(x) = Ax + B$. But

$$0 = F(0) = A \cdot 0 + B = B \qquad \text{and} \qquad 0 = F'(2\pi) = A\,,$$

thus $F = 0$. This gives us a trivial solution.

Now suppose that $\underline{K > 0}$. Then the solution of $F'' - KF = 0$ is of the form $F(x) = A\cosh(\sqrt{K}x) + B\sinh(\sqrt{K}x)$. But

$$0 = F(0) = A\cosh(0) + B\sinh(0) = A \qquad \text{and} \qquad 0 = F'(2\pi) = B\sqrt{K}\cosh(0) = B\sqrt{K} = 0\,,$$

thus $F = 0$. This gives us also a trivial solution.

Finally suppose that $\underline{K < 0}$. We write $K = -p^2$ where $p > 0$. Then the ODE $F'' + p^2F = 0$ has solution $F(x) = A\cos(px) + B\sin(px)$. But

$$0 = F(0) = A\cos 0 + B\sin 0 = A \qquad \text{and} \qquad 0 = F'(2\pi) = Bp\cos(p2\pi)\,.$$

But this means that $p2\pi = \frac{2n+1}{2}\pi$ for $n \in \mathbb{Z}$, from where it follows that

$$p = \frac{2n+1}{4} \qquad \text{with } n \in \mathbb{Z}\,.$$

Then we define the functions

$$F_n(x) = \sin\left(\frac{2n+1}{4}x\right) \qquad \text{for } n \in \mathbb{Z}\,.$$

Observe that since sin is an odd function for every $n \in \mathbb{N}$ we have that

$$F_{-n}(x) = \sin\left(\frac{-2n+1}{4}x\right) = -\sin\left(\frac{2n-1}{4}x\right) = -F_{n-1}(x)\,.$$

So the negative n's do not give us any new solutions. Then it is enough to use the functions

$$F_n(x) = \sin\left(\frac{2n+1}{4}x\right) \qquad \text{for } n = 0, 1, 2, \ldots$$

Now we are going to solve the ODE $G' - 2KG = 0$, but since $K = -p^2 = -(\frac{2n+1}{4})^2$ for $n = 0, 1, 2, \ldots$, we can write the ODE

$$G' + 2\left(\frac{2n+1}{4}\right)^2 G = 0\,,$$

that has solution $G_n(t) = e^{-2\left(\frac{2n+1}{4}\right)^2 t}$ for $n = 0, 1, 2, \ldots$.

Then our desired solutions are

$$u_n(x,t) = F_n(x)G_n(t) = \sin\left(\frac{2n+1}{4}x\right) e^{-2\left(\frac{2n+1}{4}\right)^2 t} \qquad \text{for } n = 0, 1, 2, \ldots$$

Now we want to find a linear combination $u(x,t) = \sum_{n=0}^{\infty} B_n u_n(x,t)$ such that

$$\begin{aligned} u(x,0) &= \sum_{n=0}^{\infty} B_n u_n(x,0) = \sum_{n=0}^{\infty} B_n F_n(x) G_n(0) \\ &= \sum_{n=0}^{\infty} B_n \sin\left(\frac{2n+1}{4}x\right) = \cos(x)\sin\left(\frac{x}{4}\right). \end{aligned}$$

But $\cos(x)\sin\left(\frac{x}{4}\right) = -\frac{1}{2}\sin\left(\frac{3}{4}x\right) + \frac{1}{2}\sin\left(\frac{5}{4}x\right)$, so it follows that $B_1 = -\frac{1}{2}$ and $B_2 = \frac{1}{2}$ and the rest of B_n's are zero. Therefore our desired solution is

$$u(x,t) = -\frac{1}{2}\sin\left(\frac{3}{4}x\right) e^{-2\left(\frac{3}{4}\right)^2 t} + \frac{1}{2}\sin\left(\frac{5}{4}x\right) e^{-2\left(\frac{5}{4}\right)^2 t}.$$

Problem 3

Show the Fourier transform $\mathcal{F}(x \cdot e^{-|x|}) = -\frac{2\sqrt{2}i}{\sqrt{\pi}} \frac{w}{(1+w^2)^2}$. Use this to compute

$$\int_{-\infty}^{\infty} \frac{w \sin w}{(1+w^2)^2}\, dw.$$

Possible Solution

We have that

$$\begin{aligned} \mathcal{F}(xe^{-|x|}) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xe^{-|x|} e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} xe^{x} e^{-iwx}\, dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} xe^{-x} e^{-iwx}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} xe^{(1-iw)x}\, dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} xe^{(-1-iw)x}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{xe^{(1-iw)x}}{(1-iw)} - \frac{e^{(1-iw)x}}{(1-iw)^2} \right]_{x=-\infty}^{x=0} + \frac{1}{\sqrt{2\pi}} \left(\frac{xe^{(-1-iw)x}}{(-1-iw)} - \frac{e^{(-1-iw)x}}{(-1-iw)^2} \right]_{x=0}^{x=\infty} \\ &= \frac{1}{\sqrt{2\pi}} \frac{-1}{(1-iw)^2} + \frac{1}{\sqrt{2\pi}} \frac{1}{(1+iw)^2} = \frac{1}{\sqrt{2\pi}} \frac{-(1+iw)^2 + (1-iw)^2}{(1+iw)^2(1-iw)^2} \\ &= \frac{1}{\sqrt{2\pi}} \frac{-4iw}{((1+iw)(1-iw))^2} = -\frac{2\sqrt{2}i}{\sqrt{\pi}} \frac{w}{(1+w^2)^2}, \end{aligned}$$

as desired.

For the last part, we use the inverse of the Fourier transform, that is,

$$xe^{-|x|} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -\frac{2\sqrt{2}i}{\sqrt{\pi}} \frac{w}{(1+w^2)^2} e^{iwx}\, dw = -\frac{2}{\pi} \int_{-\infty}^{\infty} i \frac{w}{(1+w^2)^2} e^{iwx}\, dw\,.$$

Now if we set $x = 1$ we have that

$$\begin{aligned} e^{-1} &= -\frac{2}{\pi} \int_{-\infty}^{\infty} i \frac{w}{(1+w^2)^2} e^{iw}\, dw = -\frac{2}{\pi} \int_{-\infty}^{\infty} i \frac{w}{(1+w^2)^2} (\cos w + i \sin w)\, dw \\ &= -i\frac{2}{\pi} \int_{-\infty}^{\infty} \frac{w}{(1+w^2)^2} \cos w\, dw + \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{w}{(1+w^2)^2} \sin w\, dw \end{aligned}$$

But then observe that $\int_{-\infty}^{\infty} \frac{w}{(1+w^2)^2} \cos w\, dw = 0$ because $\frac{w}{(1+w^2)^2} \cos w$ is an odd function, so it follows that

$$e^{-1} = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{w}{(1+w^2)^2} \sin w\, dw$$

whence

$$\frac{\pi}{2e} = \int_{-\infty}^{\infty} \frac{w}{(1+w^2)^2} \sin w\, dw\,.$$

Problem 4

Perfom 3 iterations of the Newton method to find the root of the function $f(x) = x - e^{-x}$ with $x_0 = 0$. (Use only 4 decimals in your computations).

Possible Solution

Observe that $f'(x) = 1 + e^{-x}$, then Newtons metode is the iteration given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - e^{-x_n}}{1 + e^{-x_n}} = \frac{x_n e^{-x_n} + e^{-x_n}}{1 + e^{-x_n}}\,.$$

Then we have that

$$x_0 = 0\,, \qquad x_1 = 0.5\,, \qquad x_2 = 0.5663\,, \qquad x_3 = 0.5671\,.$$

Problem 5

Use the Laplace transform to solve the differential equation

$$y'' - 3y' + 2y = 2e^{3t}\,,$$

with initial conditions

$$y(0) = 0 \qquad \text{and} \qquad y'(0) = 4\,.$$

Possible Solution

We use the Laplace transform in the ODE, and denoting Y the Laplace transform of y, we have that

$$s^2Y - sy(0) - y'(0) - 3(sY - y(0)) + 2Y = \frac{2}{s-3}\,,$$

so we have that

$$s^2Y - 4 - 3sY + 2Y = \frac{2}{s-3}\,,$$

and

$$Y(s^2 - 3s + 2) = \frac{2}{s-3} + 4 = \frac{4s-10}{s-3}\,,$$

and

$$Y = \frac{4s-10}{(s-3)(s^2-3s+2)} = \frac{4s-10}{(s-3)(s-2)(s-1)} = \frac{1}{s-3} + \frac{2}{s-2} - \frac{3}{s-1}\,.$$

So then applying the inverse of the Laplace transform we get

$$y(t) = e^{3t} + 2e^{2t} - 3e^{t}\,.$$

Problem 6

Find the polynomial of smallest degree that interpolates the points of the function $f(x)$

x_i	-2	-1	0	1	2
$f(x_i)$	1	2	5	4	1

Use this polynomial to estimate $f(3)$.

Possible Solution

Using Newton interpolation[1], we obtain

$$\begin{array}{r||lllll}
-2 & 1 & & & & \\
 & & 1 & & & \\
-1 & 2 & & 1 & & \\
 & & 3 & & -1 & \\
0 & 5 & & -2 & & 1/3 \\
 & & -1 & & 1/3 & \\
1 & 4 & & -1 & & \\
 & & -3 & & & \\
2 & 1 & & & &
\end{array}$$

[1] Lagrange interpolation would be fine as well

Then the polynomial is of the form

$$p(x) = \frac{1}{3}(x-1)x(x+1)(x+2) - x(x+1)(x+2) + (x+1)(x+2) + (x+2) + 1$$

Then the estimate of $f(3)$ can be done with $p(3) = 6$.

Problem 7

We want to numerically evaluate the integral

$$\int_0^1 f(x)\,dx \qquad \text{where} \qquad f(x) = \sin(x^2)\,,$$

with the Simpson method such that the approximation error is smaller than 0.001. What is the largest value of the step size h that this accuracy is guaranteed? Use this h to compute a numerical approximation of the above integral by the Simpson method. (Use only 4 decimals in your computations). (Hint: You can use that $\max_{0\leq x\leq 1} |f^{(4)}(x)| \leq 30$).

Possible Solution

We use the error estimation for the Simpsons methode, that says that

$$|\epsilon| \leq h^4 \frac{b-a}{180} \max_{a\leq x\leq b} |f^{(4)}(x)| \leq h^4 \cdot \frac{1-0}{180} \cdot 30 = \frac{h^4}{60}\,.$$

Since we want that $|\epsilon| < 0.001$ we may choose h such that $\frac{h^4}{6} < 0.001$, so $h < 0.2783$. Then if we want to use Simpsons method with this accuracy we need to pick n such that $\frac{1}{n} < 0.2783$. So observe that with $n = 4$, we have that $1/4 = 0.25 < 0.2783$. Then we compute

$$S_4 = \frac{0.25}{3}\Big(\sin(0) + 4\sin((0.25)^2) + 2\sin((0.5)^2) + 4\sin((0.75)^2) + \sin((1)^2)\Big) = 0.3099$$

Problem 8

Let $y(x)$ be the function that solves the ODE

$$y' = \frac{-x}{y} \qquad \text{and} \qquad y(0) = 1\,.$$

Use the Euler method with $h = 0.1$ to approximate the values of $y(x)$ at the points

$$x_1 = 0.1\,, \qquad x_2 = 0.2 \qquad \text{and} \qquad x_3 = 0.3\,.$$

(Use only 4 decimals in your computations)

Possible Solution

We have the ODE $y' = f(x,y) = \frac{-x}{y}$. Then the Euler method iteration says

$$y_{n+1} = y_n + hf(x_n, y_n)\,.$$

Then if we start with $x_0 = 0$ and $y_0 = 1$ and with $h = 0.1$, we have that

$$y_1 = 1\,, \qquad y_2 = 0.99\,, \qquad y_3 = 0.9698\,.$$

Problem 9

Write the linear system

$$\begin{aligned} 10x + y - z &= 18 \\ -x + y + 20z &= 17 \\ x + 15y + z &= -12 \end{aligned}$$

in such a form that you can apply the Gauss-Seidel method and it converges. Then perform 3 iterations with starting values $x_0 = y_0 = z_0 = 0$. (Use only 4 decimals in your computations).

Possible Solution

First we can rearrange the linear system in the following way

$$\begin{aligned} x + \frac{1}{10}y - \frac{1}{10}z &= \frac{18}{10} \\ \frac{1}{15}x + y + \frac{1}{15}z &= -\frac{12}{15} \\ -\frac{1}{20}x + \frac{1}{20}y + z &= \frac{17}{20} \end{aligned}$$

Then we can apply Gauss-Seidel to

$$A = \begin{pmatrix} 1 & \frac{1}{10} & -\frac{1}{10} \\ \frac{1}{15} & 1 & \frac{1}{15} \\ -\frac{1}{20} & \frac{1}{20} & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{15} & 0 & 0 \\ -\frac{1}{20} & \frac{1}{20} & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{10} & -\frac{1}{10} \\ 0 & 0 & \frac{1}{15} \\ 0 & 0 & 0 \end{pmatrix} = L+I+U$$

Observe that the method will converge because A is diagonal dominant.

We have then the following iteration equations

$$x^{(n+1)} = \frac{18}{10} - \frac{1}{10}y^{(n)} + \frac{1}{10}z^{(n)}$$
$$y^{(n+1)} = -\frac{12}{15} - \frac{1}{15}x^{(n+1)} - \frac{1}{15}z^{(n)}$$
$$z^{(n+1)} = \frac{17}{20} + \frac{1}{20}x^{(n+1)} - \frac{1}{20}y^{(n+1)}$$

Then if $(x^{(0)}, y^{(0)}, z^{(0)}) = (0, 0, 0)$ we have that

$$(x^{(1)}, y^{(1)}, z^{(1)}) = (1.8, -0.92, 0.986)$$
$$(x^{(2)}, y^{(2)}, z^{(2)}) = (1.9906, -0.9984, 0.9995)$$
$$(x^{(3)}, y^{(3)}, z^{(3)}) = (2, -1, 1)\,.$$

Problem 10

Let $\mathcal{R}$ be the region defined by the lines

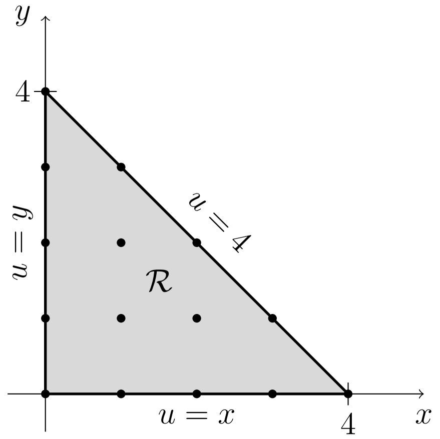
$$L_1 : \; y = 4 - x \qquad L_2 : y = 0 \qquad L_3 : x = 0\,.$$

Let $u(x, y)$ be the function defined in $\mathcal{R}$ that satisfies the Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2xy$$

and boundary conditions

$$\begin{aligned} u(x,y) &= 4 \quad \text{if } (x,y) \in L_1 \\ u(x,y) &= x \quad \text{if } (x,y) \in L_2 \\ u(x,y) &= y \quad \text{if } (x,y) \in L_3 \end{aligned}$$



Let us define the points $(x_i, y_j) = (i \cdot h, j \cdot h)$ with $h = 1$. Use the method of difference equations with $h = 1$ in order to set up a linear system for finding approximations of the values $u(1,1)$, $u(1,2)$ and $u(2,1)$.

Possible Solution

If we define $u_{i,j} := u(ih, jh)$, the difference method gives us the following equation

$$\frac{1}{h^2}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}) = 2(ih \cdot jh)$$

So if $h = 1$ we have the following equations

$$u_{2,1} + u_{0,1} + u_{1,2} + u_{1,0} - 4u_{1,1} = 2(1 \cdot 1) = 2$$

$$u_{3,1} + u_{1,1} + u_{2,2} + u_{2,0} - 4u_{2,1} = 2(2 \cdot 1) = 4$$

$$u_{2,2} + u_{0,2} + u_{1,3} + u_{1,1} - 4u_{1,2} = 2(1 \cdot 2) = 4$$

and using the boundary conditions

$$u_{2,1} + 1 + u_{1,2} + 1 - 4u_{1,1} = 2$$

$$4 + u_{1,1} + 4 + 2 - 4u_{2,1} = 4$$

$$4 + 2 + 4 + u_{1,1} - 4u_{1,2} = 4$$

Hence the the desired linear system is

$$u_{2,1} + u_{1,2} - 4u_{1,1} = 0$$

$$u_{1,1} - 4u_{2,1} = -6$$

$$u_{1,1} - 4u_{1,2} = -6$$