NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for **TMA4130 Calculus 4N**

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Examination date: December 3rd 2016

Examination time (from–to): 09:00–13:00

Permitted examination support material: Kode C:
Bestemt, enkel kalkulator
Rottmann: Matematisk formelsamling

Other information:

- All answers have to be justified, and they should include enough details in order to see how they have been obtained.
- All sub-problems carry the same weight for grading.
- Good luck!

Language: English

Number of pages: 3

Number of pages enclosed: 2

Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
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Date Signature

Problem 1 Let f be the 2π-periodic functions defined by $f(x) = \cos\left(\frac{x}{2}\right)$ when $x \in [-\pi, \pi]$. Make a drawing of the function f for the interval $[-3\pi, 3\pi]$, and compute the Fourier series of f. Use the result to compute the value of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1-4n^2} .$$

Problem 2 Find all the non-trivial solutions of the heat equation

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \qquad \text{where} \qquad 0 \le x \le 2\pi \qquad \text{and} \qquad t \ge 0 ,$$

that are of the form $u(x,t) = F(x) \cdot G(t)$, and that satisfy the boundary conditions

$$u(0,t) = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial x}(2\pi, t) = 0 \qquad \text{for every } t \ge 0 .$$

Use this to find a solution satisfying the initial condition

$$u(x,0) = \cos(x)\sin\left(\frac{x}{4}\right) \qquad \text{for every } 0 \le x \le 2\pi .$$

Problem 3 Show the Fourier transform $\mathcal{F}(x \cdot e^{-|x|}) = -\frac{2\sqrt{2}i}{\sqrt{\pi}} \frac{w}{(1+w^2)^2}$. Use this to compute

$$\int_{-\infty}^{\infty} \frac{w \sin w}{(1+w^2)^2} \, dw .$$

Problem 4 Perfom 3 iterations of the Newton method to find the root of the function $f(x) = x - e^{-x}$ with $x_0 = 0$. (Use only 4 decimals in your computations).

Problem 5 Use the Laplace transform to solve the differential equation

$$y'' - 3y' + 2y = 2e^{3t} ,$$

with initial conditions

$$y(0) = 0 \qquad \text{and} \qquad y'(0) = 4 .$$

Problem 6 Find the polynomial of smallest degree that interpolates the points of the function $f(x)$

x_i	-2	-1	0	1	2
$f(x_i)$	1	2	5	4	1

Use this polynomial to estimate $f(3)$.

Problem 7 We want to numerically evaluate the integral

$$\int_0^1 f(x)\,dx \qquad \text{where} \qquad f(x) = \sin(x^2)\,,$$

with the Simpson method such that the approximation error is smaller than 0.001. What is the largest value of the step size h that this accuracy is guaranteed? Use this h to compute a numerical approximation of the above integral by the Simpson method. (Use only 4 decimals in your computations). (Hint: You can use that $\max_{0\le x\le 1} |f^{(4)}(x)| \le 30$).

Problem 8 Let $y(x)$ be the function that solves the ODE

$$y' = \frac{-x}{y} \qquad \text{and} \qquad y(0) = 1\,.$$

Use the Euler method with $h = 0.1$ to approximate the values of $y(x)$ at the points

$$x_1 = 0.1\,, \qquad x_2 = 0.2 \qquad \text{and} \qquad x_3 = 0.3\,.$$

(Use only 4 decimals in your computations)

Problem 9 Write the linear system

$$\begin{aligned} 10x + y - z &= 18 \\ -x + y + 20z &= 17 \\ x + 15y + z &= -12 \end{aligned}$$

in such a form that you can apply the Gauss-Seidel method and it converges. Then perform 3 iterations with starting values $x_0 = y_0 = z_0 = 0$. (Use only 4 decimals in your computations).

Problem 10 Let $\mathcal{R}$ be the region defined by the lines

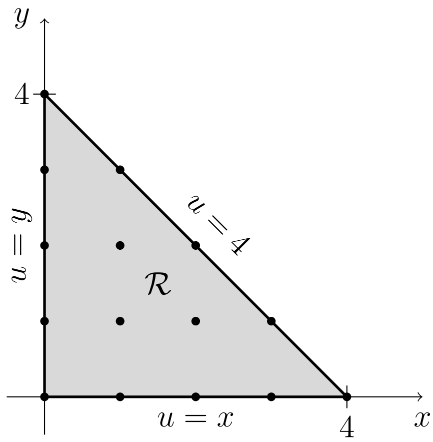
$$L_1 : \; y = 4 - x \qquad L_2 : y = 0 \qquad L_3 : x = 0\,.$$

Let $u(x, y)$ be the function defined in $\mathcal{R}$ that satisfies the Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2xy$$

and boundary conditions

$$\begin{aligned} u(x,y) &= 4 \quad \text{if } (x,y) \in L_1 \\ u(x,y) &= x \quad \text{if } (x,y) \in L_2 \\ u(x,y) &= y \quad \text{if } (x,y) \in L_3 \end{aligned}$$



Let us define the points $(x_i, y_j) = (i \cdot h, j \cdot h)$ with $h = 1$. Use the method of difference equations with $h = 1$ in order to set up a linear system for finding approximations of the values $u(1, 1)$, $u(1, 2)$ and $u(2, 1)$.

Fourier

$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\, d\omega$	$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$
$f * g(x)$	$\sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)$
$f'(x)$	$i\omega \hat{f}(\omega)$
e^{-ax^2}	$\frac{1}{\sqrt{2a}} e^{-\omega^2/4a}$
$e^{-a\|x\|}$	$\sqrt{\frac{2}{\pi}} \frac{a}{\omega^2+a^2}$
$\frac{1}{1+x^2}$	$\sqrt{\frac{\pi}{2}} e^{-\|\omega\|}$
$f(x) = 1$ for $\|x\| < a$, 0 otherwise	$\sqrt{\frac{2}{\pi}} \frac{\sin \omega a}{\omega}$

Laplace transform

$f(t)$	$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$
$f'(t)$	$sF(s) - f(0)$
$tf(t)$	$-F'(s)$
$e^{at} f(t)$	$F(s-a)$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$f(t-a)u(t-a)$	$e^{-sa} F(s)$
$\delta(t-a)$	e^{-as}
$f * g(t)$	$F(s)G(s)$

Numerics

- Newton's method: $x_{k+1} = x_k - f(x_k)/f'(x_k)$.

- Newton's method for systems: $\mathbf{J}^{(k)}\Big(\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\Big) = -\mathbf{f}(\mathbf{x}^{(k)})$ with $(\mathbf{J}^{(k)})_{ij} = \partial_j f_i^{(k)}$
- Lagrange interpolation polynomial: $L_k(x) = \frac{(x-x_0)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$, $p_n(x) = \sum_{k=0}^n L_k(x)f(x_k)$
- Trapezoid rule: $\int_a^b f(x)\,dx \approx h\Big[\frac{1}{2}f(a) + f(x_1) + f(x_2) + \ldots + f(x_{n-1}) + \frac{1}{2}f(b)\Big]$

  Error of the trapezoid rule: $|\epsilon| \leq h^2 \frac{b-a}{12} \max_{a\leq x\leq b} |f''(x)|$.
- Simpson rule: $\int_a^b f(x)\,dx \approx \frac{h}{3}\Big[f_0 + 4f_1 + 2f_2 + 4f_3 + \ldots + 2f_{n-2} + 4f_{n-1} + f_n\Big]$ with $f_i = f(x_i)$.

  Error of the Simpson rule: $|\epsilon| \leq h^4 \frac{b-a}{180} \max_{a\leq x\leq b} |f^{(4)}(x)|$.
- Gauss–Seidel iteration: $\mathbf{x}^{(k+1)} = \mathbf{b} - \mathbf{L}\mathbf{x}^{(k+1)} - \mathbf{U}\mathbf{x}^{(k)}$ with $\mathbf{A} = \mathbf{I} + \mathbf{L} + \mathbf{U}$.
- Jacobi iteration: $\mathbf{x}^{(k+1)} = \mathbf{b} + (\mathbf{I} - \mathbf{A})\mathbf{x}^{(k)}$
- Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(t_n, \mathbf{y}_n)$
- Improved Euler method: $\mathbf{k}_1 = h\mathbf{f}(t_n, \mathbf{y}_n)$, $\mathbf{k}_2 = h\mathbf{f}(t_n + h, \mathbf{y}_n + \mathbf{k}_1)$,

  $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}\mathbf{k}_1 + \frac{1}{2}\mathbf{k}_2$.
- Classical Runge–Kutte method:

  $\mathbf{k}_1 = h\mathbf{f}(t_n, \mathbf{y}_n)$, $\mathbf{k}_2 = h\mathbf{f}(t_n + h/2, \mathbf{y}_n + \mathbf{k}_1/2)$,

  $\mathbf{k}_3 = h\mathbf{f}(t_n + h/2, \mathbf{y}_n + \mathbf{k}_2/2)$, $\mathbf{k}_4 = h\mathbf{f}(t_n + h, \mathbf{y}_n + \mathbf{k}_3)$,

  $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{6}\mathbf{k}_1 + \frac{1}{3}\mathbf{k}_2 + \frac{1}{3}\mathbf{k}_3 + \frac{1}{6}\mathbf{k}_4$.
- Backward Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1})$
- Finite differences: $\frac{\partial u}{\partial x}(x,y) \approx \frac{u(x+h,y)-u(x-h,y)}{2h}$

  $\frac{\partial^2 u}{\partial x^2}(x,y) \approx \frac{u(x+h,y)-2u(x,y)+u(x-h,y)}{h^2}$

  $\frac{\partial u}{\partial y}(x,y) \approx \frac{u(x,y+h)-u(x,y-h)}{2h}$

  $\frac{\partial^2 u}{\partial y^2}(x,y) \approx \frac{u(x,y+h)-2u(x,y)+u(x,y-h)}{h^2}$