EXAMPLE 1 Complex Fourier series

Find the complex Fourier series of $f(x) = e^x$ if $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$ and obtain from it the usual Fourier series.

Solution. Since $\sin n\pi = 0$ for integer n, we have

$$e^{\pm in\pi} = \cos n\pi \pm i \sin n\pi = \cos n\pi = (-1)^n.$$

With this we obtain from (8) by integration

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx}\,dx = \frac{1}{2\pi}\frac{1}{1-in}e^{x-inx}\Big|_{x=-\pi}^{\pi} = \frac{1}{2\pi}\frac{1}{1-in}(e^{\pi} - e^{-\pi})(-1)^n.$$

On the right,

$$\frac{1}{1-in} = \frac{1+in}{(1-in)(1+in)} = \frac{1+in}{1+n^2} \quad \text{and} \quad e^{\pi} - e^{-\pi} = 2\sinh\pi.$$

Hence the complex Fourier series is

$$e^x = \frac{\sinh\pi}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\frac{1+in}{1+n^2}e^{inx} \qquad (-\pi < x < \pi). \tag{10}$$

From this let us derive the real Fourier series. Using (2) and $i^2 = -1$ we have in (10)

$$(1 + in)e^{inx} = (1 + in)(\cos nx + i \sin nx) = (\cos nx - n \sin nx) + i(n \cos nx + \sin nx).$$

Now (10) also has a corresponding term with $-n$ instead of n. Since $\cos(-nx) = \cos nx$ and $\sin(-nx) = -\sin nx$, we obtain in this term

$$(1 - in)e^{-inx} = (1 - in)(\cos nx - i \sin nx) = (\cos nx - n \sin nx) - i(n \cos nx + \sin nx).$$

If we add these two expressions, the imaginary parts cancel. Hence their sum is

$$2(\cos nx - n \sin nx), \qquad n = 1, 2, \cdots.$$

For $n = 0$ we get 1 (not 2) because there is only one term. Hence the real Fourier series is

$$e^x = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} - \frac{1}{1+1^2}(\cos x - \sin x) + \frac{1}{1+2^2}(\cos 2x - 2\sin 2x) - + \cdots\right] \tag{11}$$

where $-\pi < x < \pi$. ◀

PROBLEM SET 10.5

1. **(Calculus review)** Review complex numbers.

Complex Fourier Series. Find the complex Fourier series of the following functions. (Show the details of your work.)

2. $f(x) = -1$ if $-\pi < x < 0$, $f(x) = 1$ if $0 < x < \pi$.
3. $f(x) = x \quad (-\pi < x < \pi)$
4. $f(x) = 0$ if $-\pi < x < 0$, $f(x) = 1$ if $0 < x < \pi$
5. $f(x) = x \quad (0 < x < 2\pi)$
6. $f(x) = x^2 \quad (-\pi < x < \pi)$

7. **(Even and odd functions)** Show that the complex Fourier coefficients of an even function are real and those of an odd function are pure imaginary.
8. **(Conversion)** Convert the Fourier series in Prob. 5 to real form.
9. **(Fourier coefficients)** Show that $a_0 = c_0$, $a_n = c_n + c_{-n}$, $b_n = i(c_n - c_{-n})$, $n = 1, 2, \cdots$.
10. **PROJECT. Complex Fourier Coefficients.** It is very interesting that the c_n in (8) can be derived directly by a method similar to that for the a_n and b_n in Sec. 10.2. For this, multiply the series in (8) by e^{-imx} with fixed integer m and integrate termwise from $-\pi$ to π on both sides (allowed, for instance, in the case of uniform convergence), to get

$$\int_{-\pi}^{\pi} f(x)e^{-imx}\,dx = \sum_{n=-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{i(n-m)x}\,dx.$$

Show that the integral on the right equals 2π when $n = m$ and 0 when $n \neq m$ [use (5)], so that you get the coefficient formula in (8).

10.6 Forced Oscillations

Fourier series have important applications in differential equations. We show this for a basic problem involving an ordinary differential equation. Numerous applications to partial differential equations will follow in Chap. 11. All this will justify Euler's and Fourier's idea of splitting up a periodic function in a series of (simpler) such functions, an idea whose enormous usefulness was far from obvious.

From Sec. 2.11 we know that forced oscillations of a body of mass m on a spring of modulus k are governed by the equation

$$my'' + cy' + ky = r(t), \tag{1}$$

where $y = y(t)$ is the displacement from rest, c the damping constant, and $r(t)$ the external force depending on time t. Figure 249 shows the model and Fig. 250 its electrical analog, an RLC-circuit governed by

$$LI'' + RI' + \frac{1}{C}I = E'(t) \qquad \text{(Sec. 2.12).} \tag{1*}$$

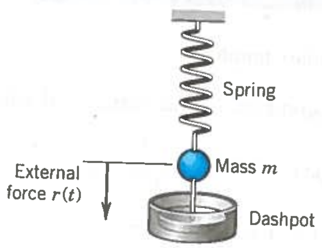

Fig. 249. Vibrating system under consideration

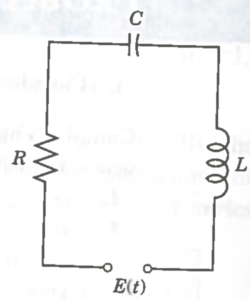

Fig. 250. Electrical analog of the system in Fig. 249 (RLC-circuit)