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Exam in TMA4123/TMA4125 Calculus 4M/4N

English
Thursday May 23, 2013
Time: 09:00 – 13:00
Grades available June 13, 2013

Examination aids (code C): Simple calculator (HP30S or Citizen SR-270X/SR-270X College.)
Rottmann: *Matematisk formelsamling.*
A two-page list of formulas attached to the exam.

Every answer must be justified and your calculations should be detailed enough to clearly show your arguments.

TMA4123 Calcuclus 4M: Everything except Problem 6 (Laplace transform)
TMA4125 Calculus 4N: Everything except Problem 7 (MATLAB)

Problem 1

a) Show that the Fourier transform of the function

$$f(x) = e^{-a|x|}, \quad -\infty < x < \infty, a > 0$$

is

$$\hat{f}(w) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + w^2}$$

directly from the definition of the Fourier transform.

b) Use the Fourier integral of the function in **a)** to calculate the integral

$$\int_0^\infty \frac{\cos wx}{w^2 + a^2}\, \mathrm{dw}\,.$$

Problem 2 Let f be the 2π-periodic function defined on the interval $[-\pi, \pi]$ by

$$f(x) = \begin{cases} 0, & -\pi \leq x \leq 0 \\ \pi - x, & 0 < x \leq \pi. \end{cases}$$

Find the *complex form* of the Fourier series of f
NB: It is not necessary to simplify your answer beyond solving the appropriate integral.

Problem 3 Consider the partial differential equation

$$u_t = u_{xx}, \qquad 0 \leq x \leq L, \quad t \geq 0 \tag{1}$$

a) Find all the solutions of (1) that can be written in the form

$$u(x,t) = F(x)G(t)$$

and satisfy the boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0.$$

b) Find the two solutions that in addition satisfy

(i) the initial condition $u(x,0) = -3\sin\left(\frac{13\pi x}{L}\right)$, where $0 \leq x \leq L$
(ii) the initial condition $u(x,0) = 10$, where $0 \leq x \leq L$,

respectively.

Problem 4

a) Find the second order polynomial $p_2(x)$ interpolating the function

$$f(x) = \frac{1}{1+x^2}$$

at the points $x = 0, 1, 2$.

b) Consider the initial value problem

$$\frac{dy}{dx} = \frac{1}{1+x^2}, \quad y(0) = 1. \tag{2}$$

Show that it can be written as the integral equation

$$y(x) = 1 + \int_0^x \frac{1}{1+t^2}\,\mathrm{dt}\,.$$

Use the result in **a)** to find an approximate solution of (2) at the point $x = 5$. Then find the exact solution of the initial value problem. What is the absolute error in the approximation?

Problem 5 Consider the following system of first order ordinary differential equations

$$\begin{aligned} x' &= -x + y, \quad x(0) = 0 \\ y' &= -x - y, \quad y(0) = 4. \end{aligned} \tag{3}$$

Use *one* iteration of Heun's method (also called the improved Euler method) with step-size $h = 0.2$ to approximate the solution. A description of Heun's method can be found in the list of formulas attached to the exam.

Problem 6 **For TMA4125/Calculus 4N**

a) Find the inverse Laplace transform of the functions

$$\frac{s-3}{(s-1)(s-4)}, \quad \frac{e^{-as}}{s(s-1)(s-4)}$$

where a is a positive constant.

b) Solve the initial value problem

$$\begin{cases} y''(t) - 5y'(t) + 4y(t) = H(t-1) - H(t-2) \\ y(0) = 1, \quad y'(0) = 2 \end{cases}$$

where H is the Heaviside step function.

Problem 7 For TMA4123/Calculus 4M

Consider the system

$$\begin{array}{rcrcrcrcr} 7x & - & 2y & + & z & + & 2w & = & 17 \\ 2x & + & 8y & + & 3z & + & w & = & 17 \\ -x & + & & & 5z & + & 2w & = & 7 \\ & & 2y & - & z & + & 4w & = & 9. \end{array}$$

and the MATLAB script

```
x=1;y=1;z=1;w=1;
for k=1:N
    x=(17/7)+(2/7)*y-(1/7)*z-(2/7)*w
    y=(17/8)-(1/4)*x-(3/8)*z-(1/8)*w
    z=(7/5)+(1/5)*x-(2/5)*w
    w=(9/4)-(1/2)*y+(1/4)*z
end
```

a) Which method is implemented in the script? Perform one iteration ($N = 1$).

b) Does the method converge?