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Examination paper for **Solutions to Matematikk 4M and 4N**

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Examination date: 21. mai 2014

Examination time (from–to): 15:00 - 19:00

Permitted examination support material: Kode C/Basic calculator. Rottmann: *Mathematical formulas*

Other information:
All answers must be argued. It should be made clear how the answers are obtained.

Language: English

Number of pages: 13

Number pages enclosed: 3

Checked by:

Date Signature

Problem 1 Consider the following function:

$$f(x) = \frac{\sin(\pi x)}{x+1},$$

a) Find a polynomial $p(x)$, of lowest possible degree, that interpolates f at the points

$$(0, f(0)), \quad (0.5, f(0.5)), \quad (1, f(1)).$$

Solution: Three points may be interpolated with a quadratic polynomial $p(x)$. Using Lagrange interpolation

$$p(x) = \frac{(x-\frac{1}{2})(x-1)}{-\frac{1}{2}} \cdot 0 + \frac{(x-0)(x-1)}{\frac{1}{2}\cdot(-\frac{1}{2})} \frac{\sin(\frac{\pi}{2})}{\frac{1}{2}+1}$$
$$+ \frac{x(x-\frac{1}{2})}{\frac{1}{2}} \cdot 0 = \frac{8}{3}x(1-x).$$

b) Use $p(x)$ to estimate the integral

$$I = \int_0^1 f(x)\ dx.$$

Solution: Since $p(x)$ is a good approximation to $f(x)$, we have that

$$I \approx \int_0^1 p(x)\ dx = \frac{8}{3}\int_0^1 x(1-x)\ dx = \frac{8}{6} - \frac{8}{9} = \frac{4}{9}.$$

c) By using $p(x)$, determine a polynomial $q(x)$ that interpolates f at the point $(1.5, f(1.5))$ in addition to the points given in *a*).

Solution: We want to determine the polynomial $q(x)$ using $p(x)$. Hence, we write $q(x)$ as a sum of $p(x)$ and a polynomial which is zero in the points given in a:

$$q(x) = p(x) + \frac{x(x-\frac{1}{2})(x-1)}{\frac{3}{2}\cdot\frac{1}{2}}C,$$

where C is to be determined such that $q(1.5) = f(1.5)$. Hence,

$$C = f(1.5) - p(1.5) = \frac{\sin\left(\frac{3\pi}{2}\right)}{\frac{3}{2}+1} - \frac{8}{3}\frac{3}{2}(1-\frac{3}{2}) = \frac{8}{5},$$

and consequently,

$$q(x) = \frac{8}{3}x(1-x) + \frac{32}{15}x\left(x-\frac{1}{2}\right)(x-1).$$

Problem 2 The following function

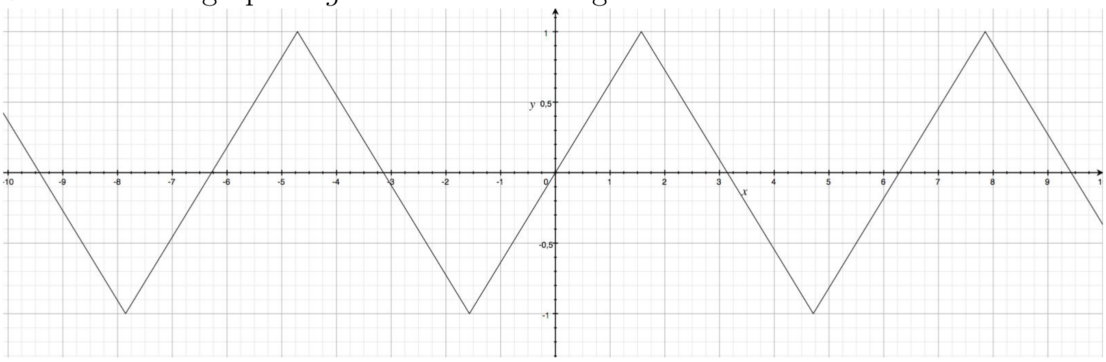
$$f(x) = \begin{cases} \frac{2x}{\pi}, & 0 \leq x < \frac{\pi}{2}, \\ \frac{2}{\pi}(\pi - x), & \frac{\pi}{2} \leq x \leq \pi, \end{cases} \tag{1}$$

is to be extended to an *odd* function g defined on $\mathbb{R}$ with period 2π.

a) Sketch the graph of g on the interval $[-3\pi, 3\pi]$ and show that g may be written

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 8}{\pi^2 (2n+1)^2} \sin\left((2n+1)x\right).$$

Solution: The graph of g takes the following form:



From the theory developed in the course, we know that $g(x)$ has the Fourierseries representation

$$g(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

where a_0, $\{a_n\}_{n>1}$, and $\{b_n\}_{n>1}$ are determined by the Euler formulas

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)\ dx, \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos(nx)\ dx, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \sin(nx)\ dx. \end{aligned}$$

Since $g(x)$ and $g(x)\cos(nx)$ are odd functions, we know that $a_0 = 0$ and $a_n = 0$, for all $n \geq 1$. Moreover, since $g(x)\sin(nx)$ is an even function

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin(nx)\ dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\ dx \\
&= \frac{4}{\pi^2}\int_{0}^{\frac{\pi}{2}} x\sin(nx)\ dx + \frac{4}{\pi^2}\int_{\frac{\pi}{2}}^{\pi} (\pi - x)\sin(nx)\ dx \\
&= \frac{4}{\pi^2}\left[-\frac{x\cos(nx)}{n}\Bigg|_{x=0}^{x=\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{\cos(nx)}{n}\ dx\right] \\
&\quad + \frac{4}{\pi^2}\left[-\frac{(\pi - x)\cos(nx)}{n}\Bigg|_{x=\frac{\pi}{2}}^{x=\pi} - \int_{\frac{\pi}{2}}^{\pi} \frac{\cos(nx)}{n}\ dx\right] \\
&= \frac{4}{\pi^2 n}\left[-\frac{\pi}{2}\cos\left(\frac{\pi}{2}n\right) + \frac{\pi}{2}\cos\left(\frac{\pi}{2}n\right)\right] + \frac{8}{\pi^2 n^2}\sin\left(n\frac{\pi}{2}\right) \\
&= \frac{8}{\pi^2 n^2}\sin\left(n\frac{\pi}{2}\right).
\end{aligned}
$$

We can further write this as

$$
b_n = \begin{cases} \frac{8}{\pi^2 n^2}, & n = 1, 5, 9, \ldots, \\ 0, & n = 2, 4, 6, 8, \ldots, , \\ -\frac{8}{\pi^2 n^2} & n = 3, 7, 11, \ldots, \end{cases}
$$

Hence, b_n is totally described by

$$
b_{2n+1} = (-1)^n \frac{8}{\pi^2 (2n+1)^2}, \quad n = 0, 1, 2, \ldots, \qquad b_{2n} = 0, \quad n = 1, 2, 3, \ldots.
$$

The Fourier series of $g(x)$ is thus given by

$$
g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 8}{\pi^2 (2n+1)^2} \sin((2n+1)x),
$$

which concludes the answer.

b) Using the Fourier series of g, prove that

$$
1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \ldots = \frac{\pi^2}{8}.
$$

Solution: By setting $x = \frac{\pi}{2}$ in the Fourierseries of $g(x)$, we find that

$$\begin{aligned} g\left(\frac{\pi}{2}\right) &= \sum_{n=0}^{\infty} \frac{(-1)^n 8}{\pi^2(2n+1)^2} \sin\left((2n+1)\frac{\pi}{2}\right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 8}{\pi^2(2n+1)^2} \sin\left(n\pi + \frac{\pi}{2}\right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 8}{\pi^2(2n+1)^2} (-1)^n = \sum_{n=0}^{\infty} \frac{8}{\pi^2(2n+1)^2}. \end{aligned}$$

Since $g(\frac{\pi}{2}) = 1$, we see that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8},$$

which is what we were asked to show.

c) An approximation of f using N Fourier components is given by

$$S^N(x) = \sum_{n=0}^{N-1} \frac{(-1)^n\, 8}{\pi^2(2n+1)^2} \sin\left((2n+1)x\right).$$

Prove that the L^2-error:

$$E = \frac{2}{\pi} \int_0^{\pi} \left|S^N(x) - f(x)\right|^2\, dx,$$

satisfies the bound

$$E < \frac{8}{\pi^2(2N+1)^2}.$$

Solution: From Parseval's identity, we have that

$$\begin{aligned} E &= \frac{2}{\pi} \int_0^{\pi} \left|S^N(x) - f(x)\right|^2\, dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \left|S^N(x) - g(x)\right|^2\, dx \\ &= \sum_{n=N}^{\infty} b_n^2 \\ &= \sum_{n=N}^{\infty} \frac{64}{\pi^4(2n+1)^4} \\ &\le \frac{64}{\pi^4} \left(\frac{1}{(2N+1)^2}\right) \sum_{n=N}^{\infty} \frac{1}{(2n+1)^2} \\ &< \frac{64}{\pi^4} \left(\frac{1}{(2N+1)^2}\right) \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}. \end{aligned}$$

From the previous exercise, we know the value of the sum on the right-hand side and hence we may conclude that

$$E < \frac{64}{\pi^4}\left(\frac{1}{(2N+1)^2}\right)\frac{\pi^2}{8} = \frac{8}{\pi^2}\frac{1}{(2N+1)^2}.$$

Problem 3 By using Fourier transform, determine the function $u(t,x)$ that satisfies the following partial differential equation:

$$u_t - u_{xx} + u = 0, \quad x \in \mathbb{R}, \quad t > 0,$$
$$u|_{t=0} = u_0, \quad x \in \mathbb{R},$$

where $u_0(x)$ is a given function.

Solution: Let $\widehat{u}(t,\omega)$ denote the Fouriertransform of $u(t,x)$ in x. That is,

$$\widehat{u}(t,\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(t,x)e^{-ix\omega}\, dx.$$

By applying Fouriertransform to the partial differential equation and using that

$$\widehat{u_{xx}} = (i\omega)^2\widehat{u} = -\omega^2\widehat{u},$$

we obtain the first order separable ordinary differential equation:

$$\widehat{u}_t + \omega^2\widehat{u} + \widehat{u} = 0.$$

The solution of this ODE is

$$\widehat{u}(t,\omega) = \widehat{u}_0(\omega)\; e^{-\omega^2 t - t}.$$

From the Formulas given at the end of the exam, we see that

$$\widehat{e^{-ax^2}} = \frac{1}{\sqrt{2a}}e^{-\frac{\omega^2}{4a}}$$

With $a = \frac{1}{4t}$, we see that

$$\frac{1}{\sqrt{2t}}\widehat{e^{-\frac{x^2}{4t}}} = e^{-\omega^2 t}$$

Hence, we can take the inverse transform to obtain that

$$u(t,x) = u_0(x) \star \left(\frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \right) e^{-t},$$

where we have used that

$$\widehat{f \star g} = \sqrt{2\pi} \widehat{f} \widehat{g},$$

which is also given in the formulas at the end of the exam.

Problem 4 Consider the following partial differential equation:

$$\begin{aligned} u_t - t u_{xx} &= 0, \quad x \in (0,1), \quad t > 0, \\ u &= 0, \quad x = 0 \text{ og } x = 1. \end{aligned}$$

a) Find all non-trivial solutions of the form $u(t,x) = G(t)F(x)$.

Solution: We begin by inserting the assertion $u(t,x) = G(t)F(x)$ in the PDE to obtain that

$$G'(t)F(x) - tG(t)F''(x) = 0.$$

By dividing by $tG(t)F(x)$, we obtain that

$$\frac{G'(t)}{tG(t)} = \frac{F''(x)}{F(x)} = k - \text{constant}.$$

Hence, we have the following two ODEs:

$$\begin{aligned} G'(t) - ktG(t) &= 0, \\ F''(x) - kF(x) &= 0. \end{aligned}$$

The second ODE will have two different types of solutions according to the sign of k. If $k > 0$, then F will be of the form $F(x) = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}$. However, this cannot satisfy the boundary conditions. That is, $F(0) = 0$ and $F(1) = 0$ can only happen if $F(x) \equiv 0$ when $k > 0$. Hence, we must have that $k < 0$.

For convenience, let us now write $k = -\lambda^2$. Then, the solutions to the above ODEs are

$$G(t) = Ce^{-\lambda^2 t^2}, \qquad F(x) = A\cos(\lambda x) + B\sin(\lambda x).$$

To satisfy boundary conditions, we need that $F(0) = 0$ and $F(1) = 0$. For the first requirement, we see that

$$F(0) = A \quad \Rightarrow \quad A = 0.$$

For the second requirement, we see that

$$F(1) = B\sin(\lambda) \quad \Rightarrow \quad \lambda = n\pi.$$

Hence, for any choice of constants C and B and any integer n, we have found that

$$G(t)F(x) = CBe^{-n^2\pi^2t^2}\sin(nx),$$

solves the PDE with the given boundary conditions. Hence, all non-trivial solution can be obtained by summing over all possible choices of constants C, B, and frequencies n yielding

$$u(t, x) = \sum_{n=0}^{\infty} B_n e^{-(n\pi)^2t^2}\sin(nx).$$

(Since C and B only appear as CB this only provides one degree of freedom which we denote by B_n).

b) Determine the unique solution $u(t, x)$ satisfying the initial condition:

$$u|_{t=0} = 2\sin(\pi x) + \frac{5}{3}\sin(30\pi x) + \frac{1}{7}\sin(11\pi x).$$

Solution: We are now to determine the coefficients B_n in $u(t, x)$ to satisfy the given initial value. That is, find B_n such that

$$u(0, x) = \sum_{n=0}^{\infty} B_n \sin(nx) = 2\sin(\pi x) + \frac{5}{3}\sin(30\pi x) + \frac{1}{7}\sin(11\pi x).$$

We readily see that,

$$B_1 = 2, \quad B_{30} = \frac{5}{3}, \quad B_{11} = \frac{1}{7}, \quad B_n = 0, \quad \forall n \neq 1, 11, 30.$$

Hence, the solution $u(t, x)$ is given by

$$u(t, x) = 2e^{-\pi^2t^2}\sin(\pi x) + \frac{1}{7}e^{-(11)^2\pi^2t^2}\sin(11\pi x) + \frac{5}{3}e^{-(30)^2\pi^2t^2}\sin(30\pi x).$$

Problem 5

Formulate Jacobi's method for solving the linear system

$$\begin{pmatrix} 3 & \frac{1}{2} & 1 \\ \frac{1}{3} & 2 & 0 \\ 1 & \frac{1}{5} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

and perform two iterations with starting vector of your own choice. Prove that the iteration will converge.

Solution: Jacobi's method for the given linear system is

$$\begin{aligned} x^{k+1} &= \frac{2}{3} - \frac{1}{6}y^k - \frac{1}{3}z^k, \\ y^{k+1} &= \frac{1}{2} - \frac{1}{6}x^k, \\ z^{k+1} &= -\frac{1}{5}y^k - x^k. \end{aligned}$$

Starting with $(x^0, y^0, z^0) = (0, 0, 0)$, we obtain

$$x^1 = \frac{2}{3}, \quad y^1 = \frac{1}{2}, \quad z^1 = 0.$$

and

$$\begin{aligned} x^2 &= \frac{2}{3} - \frac{1}{6} \cdot \frac{1}{2} - \frac{1}{3} \cdot 0 = \frac{7}{12} \\ y^2 &= \frac{1}{2} - \frac{1}{6} \cdot \frac{2}{3} = \frac{7}{18} \\ z^2 &= -\frac{1}{5} \cdot \frac{1}{2} - \frac{2}{3} = \frac{23}{30} \end{aligned}$$

Next, we turn to the issue of proving convergence of the method. This was by far the most difficult question in the exam and you should not despair if you could not do it. The primary difficulty lies in the matrix not being diagonal dominant.

We begin by defining the following matrix

$$A = \begin{pmatrix} 0 & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{6} & 0 & 0 \\ 1 & \frac{1}{5} & 0 \end{pmatrix} \tag{2}$$

and the vectors $\bar{x}^k = \left[x^k, y^k, z^k\right]^T$ and $\bar{b} = \left[\frac{2}{3}, \frac{1}{2}, 0\right]^T$. Then, the Jacobi method may be written

$$\bar{x}^{k+1} = \bar{b} - A\bar{x}^k.$$

Let us then estimate $\bar{x}^{k+1} - \bar{x}^k$. Using the method, we see that

$$\bar{x}^{k+1} - \bar{x}^k = A \cdot (\bar{x}^{k-1} - \bar{x}^k).$$

By iteratively applying the previous identity, we find that

$$\bar{x}^{k+1} - \bar{x}^k = A \cdot (\bar{x}^{k-1} - \bar{x}^k) = -A^2(\bar{x}^{k-1} - \bar{x}^{k-2}) = (-1)^k A^k \cdot (\bar{x}^1 - \bar{x}^0).$$

Hence, we have that

$$|\bar{x}^{k+1} - \bar{x}^k| \leq \|A\|^k |\bar{x}^1 - \bar{x}^0|, \qquad k \geq 2,$$

where $\|A\|$ denotes the maximal eigenvalue (in absolute value) of A. As a consequence, we can conclude that the method converges provided the spectral radius $\|A\| < 1$. The eigenvalues of A is given as the zeros of the polynomial

$$p(\lambda) = \det(A - \lambda I) = -\lambda^3 + \frac{13}{36}\lambda + \frac{1}{90}.$$

Hence, it remains to prove that any $\bar{\lambda}$ satisfying

$$p(\bar{\lambda}) = 0,$$

also satisfies $|\bar{\lambda}| < 1$. By differentiating with respect to λ, we see that

$$p'(\lambda) = -3\lambda^2 + \frac{13}{36},$$

and hence the local minima and maxima of $p(\lambda)$ is located at

$$\lambda = \pm\sqrt{\frac{13}{108}} < 1.$$

Moreover, since $p''(\lambda) = -6\lambda$, $\lambda = \sqrt{\frac{13}{108}}$ is a local maxima and $\lambda = -\sqrt{\frac{13}{108}}$ is a local minima. This means that $p(\lambda)$ is strictly decreasing for $\lambda > \sqrt{\frac{13}{108}}$ and strictly increasing for $\lambda < -\sqrt{\frac{13}{108}}$.

So are the zeros of $p(\lambda)$ in absolute value less than 1? To check this, we first calculate

$$p(1) = -1 + \frac{13}{36} + \frac{1}{90} = -\frac{2034}{3240} < 0.$$

and

$$p(-1) = 1 - \frac{13}{36} + \frac{1}{90} = \frac{2160}{3240} > 0.$$

Since $p(\lambda)$ is strictly decreasing on the interval $\lambda > 1$ and $p(1) < 0$, all zeros of $p(\lambda)$ satisfies $\lambda < 1$. On the other hand, we know that $p(\lambda)$ is strictly increasing as λ becomes more negative than $\lambda = -1$. Since $p(-1) > 0$, this means that there can be no zeros of $p(\lambda)$ with $\lambda \leq -1$. Hence, all three zeros of $p(\lambda)$ must be strictly in the interval $(-1, 1)$ which means that all eigenvalues of A have absolute value strictly less than 1 and hence the method converges.

Problem 6 In this assignment, we want to determine a numerical approximation to the solution of the following wave equation:

$$\begin{aligned} u_{tt} - 2u_{xx} &= \sin(t), \quad x \in (0,1),\ t > 0, \\ u(t,0) &= 0, \quad t > 0, \\ u(t,1) &= 0, \quad t > 0, \\ u|_{t=0} &= 0, \quad x \in (0,1), \\ u_t|_{t=0} &= 1, \quad x \in (0,1). \end{aligned} \tag{3}$$

Formulate a method that provides a numerical approximation to the solution u. What is the order of your method?

Solution: We discretize the domain $(0, 1)$ into N points with spacing h. That is, through the points

$$x_i = ih, \quad i = 0, \ldots, N-1, \quad h = \frac{1}{N-1}.$$

Next, we discretize time through the points

$$t^n = n\Delta t, \quad n = 0, 1, 2, 3, \ldots$$

where Δt is an undetermined step-size. Next, we will approximate $u(t, x)$ at these points using the notation

$$u(t^n, x_i) \approx u_i^n, \quad i = 0, \ldots, N-1,\ n = 0, 1, 2, 3, \ldots.$$

To approximate the derivatives in the PDE, we shall use the central differences

$$u_{tt}(t^n, x_i) \approx \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2},$$
$$u_{xx}(t^n, x_i) \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}.$$

In this way, we obtain the following method to determine the approximation u_i^n of u:

1. Set the initial u_i^0 and u_i^1 as follows:

$$u_i^0 = 0, \quad i = 0, \dots, N-1,$$
$$u_i^1 = \Delta t, \quad i = 0, \dots, N-1.$$

2. Determine sequentially u_i^{n+1}, $i = 0, \dots, N-1$, as follows:

   - For $i = 1, \dots, N-2$,

$$u_i^{n+1} = 2u_i^n - u_i^{n-1} + 2\frac{(\Delta t)^2}{h^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) + (\Delta t)^2 \sin(n\Delta t),$$

   - At the boundary, set

$$u_0^{n+1} = 0, \quad u_{N-1}^{n+1} = 0.$$

This method is second-order in Δt and h.

Problem 7 Use Laplace-transform to determine the function $y(t)$ satisfying the following ordinary differential equations:

a) $y'' - y' - 6y = 0, \quad y(0) = 6, \; y'(0) = 13.$

Solution: By taking the Laplace transform, we obtain

$$(s^2 - s - 6)Y(s) = (s-1)6 + 13 = 6s + 7.$$

A simple division and factorization then provides

$$Y(s) = \frac{6s + 7}{(s-3)(s+2)} = \frac{5}{s-3} + \frac{1}{s+2}.$$

Hence, we see that

$$y(t) = 5e^{3t} + e^{-2t},$$

solves the ODE.

b) $y'' + 3y' - 4y = \delta(t-2), \quad y'(0) = y(0) = 0.$

Solution: By taking the Laplace transform, we obtain

$$(s^2 + 3s - 4)Y(s) = e^{-2s}$$

Dividing and factorizing provides

$$Y(s) = \frac{e^{-2s}}{(s-1)(s+4)} = \frac{e^{-2s}}{5(s-1)} - \frac{e^{-2s}}{5(s+4)}.$$

This, we recognize as the inverse of

$$y(t) = \frac{1}{5}\left(e^{t-2} - e^{-4(t-2)}\right)\mathcal{H}(t-2),$$

where $\mathcal{H}$ is the heaviside function.

Problem 8 A signal is stored in the vector $\mathbf{y} = [y_0, y_1, \dots, y_{N-1}]$, where y_i are real numbers. $\hat{\mathbf{y}} = \text{fft}(\mathbf{y})$ is such that

$$y_k = \sum_{n=0}^{N-1} \frac{1}{N}\hat{y}_n e^{inx_k},$$

where $x_k = \frac{2\pi k}{N}$.

a) The signal represents a sound signal that is T seconds long. What is the frequency of the component described by $\frac{1}{N}\hat{y}_n e^{inx_k}$? You do not have to consider aliasing in this task.

Solution: The component $\frac{1}{N}\hat{y}_n e^{inx_k}$ is an interpolant for the function $\frac{1}{N}\hat{y}_n e^{inx}$. This function completes n periods when x goes from 0 to N. Thus, the frequency is $f_n = \frac{n}{T}$.

b) The following matlab-function takes a vector `y` and a positive integer `k`($<N/2$) as input, and returns a vector `ynew`. Given that `y` represents a sound signal that is T seconds long, describe `ynew`.

```
function ynew = mystery(y,k)
  N = numel(y); % number of elements in y.
  z = fft(y);
  znew = zeros(N,1);
  for i = 1:k
    znew(i) = z(i);
  end
  znew(N:-1:floor(N/2)+2) = conj(znew(2:ceil(N/2)));
  % ensures that ynew is real
  ynew = ifft(znew);
end
```

Solution: The function computes the Fourier transform $z = \text{fft}(y)$. Then it copies the first k components of z to $znew$, enforces the symmetry and calculates $ynew = \text{fft}^{-1}(znew)$

This corresponds to copying the k lowest frequencies from y to $ynew$. (This operation is known as a high-pass filter.) $ynew$ is a sound signal of the same length as y and consists of the frequency components with $f < f_k = \frac{k}{T}$.

Numerical formulas

- Let $p(x)$ be a polynomial of degree $\leq n$ interpolating a function $f(x)$ at the points $x_i \in [a,b], i = 0, 1, \dots, n$. For any $x \in [a,b]$,

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i).$$

- Newton's divided differences interpolation $p(x)$ of degree $\leq n$:

$$\begin{aligned} p(x) = &f[x_0] + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] \\ &+ \cdots + (x - x_0)(x - x_1) \dots (x - x_{n-1}) f[x_0, \dots, x_n] \end{aligned}$$

- Numerical differentiation:

$$\begin{aligned} f'(x) &= \frac{1}{h}\Big[f(x+h) - f(x)\Big] + \frac{1}{2} h f''(\xi) \\ f'(x) &= \frac{1}{2h}\Big[f(x+h) - f(x-h)\Big] - \frac{1}{6} h^2 f'''(\xi) \\ f''(x) &= \frac{1}{h^2}\Big[f(x+h) - 2f(x) + f(x-h)\Big] - \frac{1}{12} h^2 f^{(4)}(\xi) \end{aligned}$$

- Simpson's integration rule:

$$\int_{x_0}^{x_2} f(x)\, dx \approx \frac{h}{3}(f_0 + 4f_1 + f_2)$$

- Iterative techniques for solving linear systems:

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \qquad i = 1, 2, \dots, n$$

$$\text{Jacobi:} \quad x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right)$$

$$\text{Gauss–Seidel:} \quad x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right)$$

- Heun's method for solving $y' = f(t, y)$

$$\begin{aligned} k_1 &= h f(t_n, y_n) \\ k_2 &= h f(t_n + h, y_n + k_1) \\ y_{n+1} &= y_n + \frac{1}{2}(k_1 + k_2) \end{aligned}$$

Some Laplace transforms

$f(t)$	$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$t^n \ (n = 0, 1, 2, \ldots)$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$\cosh at$	$\frac{s}{s^2 - a^2}$
$\sinh at$	$\frac{a}{s^2 - a^2}$
$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2 + \omega^2}$
$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2 + \omega^2}$

Some Fourier transforms

$f(x)$	$\hat{f}(w) = \mathcal{F}(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx$
$g(x) = f(ax)$	$\hat{g}(w) = \frac{1}{a} \hat{f}\left(\frac{w}{a}\right)$
$u(x) - u(x-a)$	$\frac{1}{\sqrt{2\pi}} \left(\frac{\sin aw}{w} - i \frac{1 - \cos aw}{w} \right)$
$u(x)e^{-x}$	$\frac{1}{\sqrt{2\pi}} \left(\frac{1}{1+w^2} - i \frac{w}{1+w^2} \right)$
e^{-ax^2}	$\frac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}}$
$e^{-a\|x\|}$	$\sqrt{\frac{2}{\pi}} \frac{a}{w^2 + a^2}$