TMA4122 Mathematics 4M, fall 2014

Solution suggestions

1a First we must observe that $f(x)$ is an even function, so the the Fourier series of $f(x)$ will be of the form

$$a_0 + \sum_{i=1}^{\infty} a_n \cos(nx).$$

So the computation follows to

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \sin(x)dx = \left.\frac{-\cos(x)}{\pi}\right]_0^{\pi} = \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin((n+1)x) - \sin((n-1)x)\,dx$$

First if $n = 1$, we have that

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin(2x) - \sin(0x)\,dx = \left.\frac{-\cos(2x)}{2\pi}\right]_0^{\pi} = 0$$

and for $n \geq 2$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin((n+1)x) - \sin((n-1)x)\,dx = \left.\frac{1}{2\pi}\left(\frac{-\cos((n+1)x)}{n+1} + \frac{\cos((n-1)x)}{n-1}\right)\right]_0^{\pi}$$

$$= \begin{cases} \frac{-4}{\pi}\frac{1}{n^2-1} & \text{if n is even} \\ 0 & \text{if n is off} \end{cases}$$

Therefore we have that the Fourier series of $f(x)$ is

$$\frac{2}{\pi} - \frac{4}{\pi}\sum_{\text{n even}} \frac{1}{n^2-1}\cos(nx) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1}\cos(2nx)$$

1b First observe that $f(x)$ is continuous at 0, so we have that

$$0 = f(0) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos(2n0) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1},$$

therefore

$$\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}$$

2

$$\mathscr{F}(f)(w) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} f(x)e^{-iwx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} |x|(\cos(wx) - i\sin(wx))\,dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}|x|\cos(wx)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^1 x\cos(wx)\,dx$$

If $w = 0$ we have that

$$\mathscr{F}(f)(0) = \sqrt{\frac{2}{\pi}}\int_0^1 x\cos(0x)\,dx = \sqrt{\frac{2}{\pi}}\int_0^1 x\,dx = \sqrt{\frac{2}{\pi}}\left.\frac{x^2}{2}\right]_0^1 = \frac{1}{\sqrt{2\pi}},$$

and finally if $w \neq 0$

$$\mathscr{F}(f)(w) = \sqrt{\frac{2}{\pi}}\int_0^1 x\cos(wx)\,dx = \sqrt{\frac{2}{\pi}}\left.\left(\frac{x\sin(wx)}{w} + \frac{\cos(wx)}{w^2}\right)\right]_0^1 = \sqrt{\frac{2}{\pi}}\left(\frac{\cos(w)}{w^2} + \frac{\sin(w)}{w} - \frac{1}{w^2}\right)$$

3 We have that $x_0 = 0$, $y_0 = 1$ and $h = 0.1$, the applying the Euler method we have

$$x_1 = 0.1 \qquad \text{and} \qquad y_1 = 1 + 0.1 * (1 * e^0) = 1.1$$

$$x_2 = 0.2 \qquad \text{and} \qquad y_2 = 1.1 + 0.1 * (1.1 * e^{0.1}) = 1.2216$$

$$x_3 = 0.3 \qquad \text{and} \qquad y_3 = 1.2216 + 0.1 * (1.2216 * e^{0.2}) = 1.3708$$

4a Since $u = F(x)G(t)$ we have that $u_{xx} = F''G$ and $u_t = FG'$, and replacing them to the equation we have $FG' = 2F''G$. Then since

$$\frac{G'}{G} = 2\frac{F''}{F} = k \qquad \text{where k is a constant.}$$

then we can deduce 2 ODE

$$\begin{cases} F'' - \frac{k}{2}F = 0 \\ G' - kG = 0 \end{cases}$$

The boundary conditions say that $F'(0) = 0$ and $F(\pi) = 0$, so we can easily check that if $k = 0$ or $k > 0$ then $F = 0$, and hence $u = 0$. Since we want only non-trivial solutions, we neglect these options.

So we can consider $k = -p^2 < 0$, and hence the solutions to the ODE $F'' + \frac{p^2}{2}F = 0$ are of the form

$$F(x) = A\cos\left(\frac{p}{\sqrt{2}}x\right) + B\sin\left(\frac{p}{\sqrt{2}}x\right).$$

Since $F'(0) = -A\frac{p}{\sqrt{2}}\sin\left(\frac{p}{\sqrt{2}}0\right) + B\frac{p}{\sqrt{2}}\cos\left(\frac{p}{\sqrt{2}}0\right) = B\frac{p}{\sqrt{2}} = 0$, and hence $B = 0$. Finally since $F(\pi) = A\cos\left(\frac{p}{\sqrt{2}}\pi\right) = 0$, this implies $\cos\left(\frac{p}{\sqrt{2}}\pi\right) = 0$, so $\frac{p}{\sqrt{2}}\pi = \frac{2n+1}{2}\pi$ for $n = 0, 1, \ldots$. So if we isolate p we have $p = \frac{2n+1}{\sqrt{2}}$ for $n = 0, 1, \ldots$, so we have that

$$F_n(x) = \cos\left(\frac{2n+1}{2}x\right) \qquad n = 0, 1, \ldots$$

So we solve the second ODE, so $G' + p^2G = 0$ and replacing $p = \frac{2n+1}{\sqrt{2}}$ we have that

$$G_n' + \frac{(2n+1)^2}{2}G_n = 0 \qquad n = 0, 1, \ldots$$

and easily we can find the solution

$$G_n = A_n e^{-\frac{(2n+1)^2}{2}t} \qquad n = 0, 1, \ldots$$

Therefore melting both solutions we have that

$$u_n = F_nG_n = A_n e^{-\frac{(2n+1)^2}{2}t}\cos\left(\frac{2n+1}{2}x\right) \qquad n = 0, 1, \ldots$$

4b The initial conditions say that $u(x,0) = \cos(x)\cos(\frac{3}{2}x) = \frac{1}{2}\left(\cos(\frac{1}{2}x) + \cos(\frac{5}{2}x)\right)$, so

$$u(x,0) = \sum_{n=0}^{\infty} u_n(x,0) = \sum_{n=0}^{\infty} F_n(x)G_n(0) = \sum_{n=0}^{\infty} A_n\cos\left(\frac{2n+1}{2}x\right).$$

Therefore $A_0 = A_2 = \frac{1}{2}$, and hence

$$u(x,t) = \frac{1}{2}e^{-\frac{1}{2}t}\cos\left(\frac{1}{2}x\right) + \frac{1}{2}e^{-\frac{25}{2}t}\cos\left(\frac{5}{2}x\right)$$

5a We use the Lagrange polynomial formula

$$p_4(x) = 0\cdot\frac{(x+1)x(x-1)(x-2)}{(-2+1)(-2)(-2-1)(-2-2)} + 2\cdot\frac{(x+2)x(x-1)(x-2)}{(-1+2)(-1)(-1-1)(-1-2)} + 0\cdot\frac{(x+2)(x-1)(x+1)(x-2)}{(0+2)(0-1)(0+1)(0-2)}$$
$$+1\cdot\frac{(x+2)(x+1)x(x-2)}{(1+2)(1+1)1(1-2)} + 0\cdot\frac{(x+2)(x+1)x(x-1)}{(2+2)(2+1)2(2-1)}$$
$$= -\frac{1}{3}(x+2)x(x-1)(x-2) - \frac{1}{6}(x+2)(x+1)x(x-2) = -\frac{1}{2}x^4 + \frac{1}{6}x^3 + 2x^2 - \frac{2}{3}x$$

5b First remember that $h = \frac{(3-1)}{n} = \frac{2}{n}$. Using the Error formula of the Simpson method we have the following inequality

$$|E_n| \le \frac{(3-1)^5}{180n^4}\max_{[1,3]}|f^{iv)}(x)| < 0.001$$

We have that

$$f(x) = x^2\ln(x) \qquad f'(x) = 2x\ln(x) + x \qquad f''(x) = 2\ln(x) + 3 \qquad f'''(x) = \frac{2}{x} \qquad \text{and} \qquad f^{iv)}(x) = -\frac{2}{x^2}$$

so we have that $\max_{[1,3]}|-\frac{2}{x^2}| = 2$. Therefore

$$\frac{(3-1)^5}{180n^4}2 < 0.001 \qquad \text{and} \qquad n^4 > \frac{64000}{180} \qquad \text{and} \qquad n > 4.3424,$$

and hence $h = \frac{3-1}{4.3425} = 0.4606$.

To apply Simpson method, n must be an even natural number, so $n = 6$. Finally applying the Simpson method formula with $h = \frac{(3-1)}{6} = \frac{1}{3}$ we have that

$$\int_1^3 x^2\ln(x)dx \approx \frac{\frac{1}{3}}{3}\left(1^2\ln(1) + 4\left(\frac{4}{3}\right)^2\ln\left(\frac{4}{3}\right) + 2\left(\frac{5}{3}\right)^2\ln\left(\frac{5}{3}\right) + 4\left(\frac{6}{3}\right)^2\ln\left(\frac{6}{3}\right) + 2\left(\frac{7}{3}\right)^2\ln\left(\frac{7}{3}\right) + 4\left(\frac{8}{3}\right)^2\ln\left(\frac{8}{3}\right) + 3^2\ln(3)\right).$$
$$\approx 6.9985$$

6a Using the notation of the text, we have that $(x_1, y_1) = (1,1)$, $(x_2, y_1) = (2,1)$ and $(x_1, y_2) = (1,2)$, so we write $u_{11} = u(1,1)$, $u_{12} = u(1,2)$ and $u_{21} = u(2,1)$. The equation gives us the following relations

$$\frac{u_{i+1,j} + u_{i-1,j} - 2u_{i,j}}{h^2} + \frac{u_{i,j+1} + u_{i,j-1} - 2u_{i,j}}{h^2} = 2(ih)(jh)$$

so if we replace $h = 1$ we have

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 2ij$$

Therefore we set up the 3 equations with $(i,j) = (1,1)$ and $(i,j) = (1,2)$ and $(i,j) = (2,1)$

$$\begin{cases} u_{21} + u_{01} + u_{12} + u_{10} - 4u_{11} = 2 \\ u_{22} + u_{02} + u_{13} + u_{11} - 4u_{12} = 4 \\ u_{31} + u_{11} + u_{22} + u_{20} - 4u_{21} = 4 \end{cases}$$

So using the boundary conditions we have $u_{0,1} = u_{1,0} = u_{0,2} = u_{2,0} = 1$, and $u_{2,2} = \frac{2^2+2^2}{16} = \frac{1}{2}$, $u_{1,3} = \frac{1+3^2}{16} = \frac{5}{8}$ and $u_{3,1} = \frac{3^2+1}{16} = \frac{5}{8}$, so the desired system is

$$\begin{cases} u_{21} + 1 + u_{12} + 1 - 4u_{11} = 2 \\ \frac{1}{2} + 1 + \frac{5}{8} + u_{11} - 4u_{12} = 4 \\ \frac{5}{8} + u_{11} + \frac{1}{2} + 1 - 4u_{21} = 4 \end{cases}$$

and

$$\begin{cases} u_{21} + u_{12} - 4u_{11} = 0 \\ u_{11} - 4u_{12} = \frac{15}{8} \\ u_{11} - 4u_{21} = \frac{15}{8} \end{cases}$$

6b We set up the system

$$\begin{cases} u_{21}+u_{12}-4u_{11}=0 \\ u_{11}-4u_{12}=\frac{15}{8} \\ u_{11}-4u_{21}=\frac{15}{8} \end{cases}$$

in the Gauss-Seidel form

$$\begin{cases} -\frac{1}{4}u_{21}-\frac{1}{4}u_{12}+u_{11}=0 \\ -\frac{1}{4}u_{11}+u_{12}=-\frac{15}{32} \\ -\frac{1}{4}u_{11}+u_{21}=-\frac{15}{32} \end{cases}$$

so the Gauss-Seidel iteration formula is

$$\begin{aligned} u_{11}^{(n+1)} &= \frac{1}{4}u_{21}^{(n)}+\frac{1}{4}u_{12}^{(n)} \\ u_{12}^{(n+1)} &= \frac{1}{4}u_{11}^{(n+1)}-\frac{15}{32} \\ u_{21}^{(n+1)} &= \frac{1}{4}u_{11}^{(n+1)}-\frac{15}{32} \end{aligned}$$

So if we start with $u_{11}^{(0)}=u_{12}^{(0)}=u_{21}^{(0)}=0$

$$\begin{aligned} u_{11}^{(1)} &= \frac{1}{4}0+\frac{1}{4}0=0 \\ u_{12}^{(1)} &= \frac{1}{4}0-\frac{15}{32}=-\frac{15}{32} \\ u_{21}^{(1)} &= \frac{1}{4}0-\frac{15}{32}=-\frac{15}{32} \end{aligned}$$

and continue with the second step

$$\begin{aligned} u_{11}^{(2)} &= \frac{1}{4}\left(-\frac{15}{32}\right)+\frac{1}{4}\left(-\frac{15}{32}\right)=-\frac{15}{64} \\ u_{12}^{(2)} &= \frac{1}{4}\left(-\frac{15}{64}\right)-\frac{15}{32}=-\frac{135}{256} \\ u_{21}^{(2)} &= \frac{1}{4}\left(-\frac{15}{64}\right)-\frac{15}{32}=-\frac{135}{256} \end{aligned}$$

and finally the

$$\begin{aligned} u_{11}^{(3)} &= \frac{1}{4}\left(-\frac{135}{256}\right)+\frac{1}{4}\left(-\frac{135}{256}\right)=-\frac{135}{512} \\ u_{12}^{(3)} &= \frac{1}{4}\left(-\frac{135}{512}\right)-\frac{15}{32}=-\frac{1095}{2048} \\ u_{21}^{(3)} &= \frac{1}{4}\left(-\frac{135}{512}\right)-\frac{15}{32}=-\frac{1095}{2048} \end{aligned}$$