NTNU – Trondheim
Norwegian University of
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Department of Mathematical Sciences

Examination paper for **TMA4122 Matematikk 4M**

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Examination date: 2. desember 2014

Examination time (from–to): 09:00–13:00

Permitted examination support material: (Code C): Allowed a simple calculator. Rottmann: Collection of mathematical formulas

Other information:
All the answers must be justified and it should be clear how they are obtained. All the 10 subproblems of the exam have the same weight for the computation of the final mark.

Language: English

Number of pages: 3

Number pages enclosed: 2

Checked by:

Date Signature

Problem 1 Let $f(x)$ the 2π-periodic function defined by

$$f(x) = \begin{cases} -\sin(x) & \text{if } -\pi < x \leq 0, \\ \sin(x) & \text{if } 0 < x \leq \pi. \end{cases}$$

a) Compute the Fourier series of the function $f(x)$.

b) Use the Fourier series of the function $f(x)$ computed in exercise **a)** to find the value of the serie

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$$

Problem 2 Let $f(x)$ the function defined by

$$f(x) = \begin{cases} |x| & \text{if } -1 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Compute the Fourier Transform of the function $f(x)$.

Problem 3 Let $y(x)$ be the function that solves the ODE

$$y' = ye^x \qquad \text{and} \qquad y(0) = 1\,.$$

Use the Euler method with $h = 0.1$ to approximate the values of $y(x)$ at the points

$$x_1 = 0.1\,, \qquad x_2 = 0.2 \qquad \text{and} \qquad x_3 = 0.3\,.$$

(Use only 4 decimals in your computations.)

Problem 4

a) Find all the non-trivial solutions of the heat equation

$$\frac{\partial u}{\partial t} = 2 \cdot \frac{\partial^2 u}{\partial x^2} \qquad \text{where} \qquad 0 \leq x \leq \pi \qquad \text{and} \qquad t \geq 0$$

that are of the form $u(x,t) = F(x) \cdot G(t)$, and that satisfy the boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 0 \qquad \text{and} \qquad u(\pi, t) = 0 \qquad \text{for } t \geq 0.$$

b) With the results from **a)** find the solutions that additionally satisfy the initial condition

$$u(x,0) = \cos(x)\cos\left(\frac{3}{2}\,x\right) \qquad \text{for} \qquad 0 \leq x \leq \pi\,.$$

Problem 5

a) Find the polynomial of lowest degree that interpolate the points

x_i	-2	-1	0	1	2
$f(x_i)$	0	2	0	1	0

b) We want to numerically evaluate the integral

$$\int_1^3 x^2 \ln(x)dx$$

with the Simpson method such that the approximation error is smaller than 0.001. What is the largest value of the step size h that we can choose such that this accuracy is guaranteed? Use this h to compute a numerical approximation of the above integral by the Simpson method. (Use only 4 decimals in your computations)

Problem 6 Let $\mathcal{R}$ be the region defined by the lines

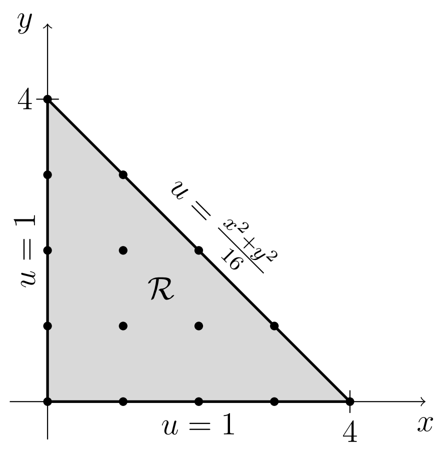
$$L_1 : y = 4 - x \qquad L_2 : y = 0 \qquad L_3 : x = 0\,.$$

Let $u(x, y)$ be the function defined in $\mathcal{R}$ satisfying the Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2xy$$

and the boundary conditions

$$\begin{array}{ll} u(x,y) = \frac{x^2+y^2}{16} & \text{if } (x,y) \in L_1 \\ u(x,y) = 1 & \text{if } (x,y) \in L_2 \\ u(x,y) = 1 & \text{if } (x,y) \in L_3 \end{array}$$



a) Let us define the points $(x_i, y_j) = (i \cdot h, j \cdot h)$ with $h = 1$. Use the method of difference equations with $h = 1$ in order to set up a linear system for finding approximations of the values $u(1, 1)$, $u(1, 2)$ and $u(2, 1)$.

b) Write the linear system you have found in **a)** in such a form that you can apply the Gauss-Seidel method. Then carry out three iterations of the Gauss-Seidel method with starting values 0 in all the variables. (Use only 4 decimals in your computations.)

Fourier

$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\, d\omega$	$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$
$f * g(x)$	$\sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)$
$f'(x)$	$i\omega\hat{f}(\omega)$
e^{-ax^2}	$\frac{1}{\sqrt{2a}} e^{-\omega^2/4a}$
$e^{-a\lvert x\rvert}$	$\sqrt{\frac{2}{\pi}} \frac{a}{\omega^2+a^2}$
$\frac{1}{1+x^2}$	$\sqrt{\frac{\pi}{2}} e^{-\lvert\omega\rvert}$
$f(x) = 1$ for $\lvert x\rvert < a$, 0 otherwise	$\sqrt{\frac{2}{\pi}} \frac{\sin \omega a}{a}$

Laplace transform

$f(t)$	$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$
$f'(t)$	$sF(s) - f(0)$
$tf(t)$	$-F'(s)$
$e^{at} f(t)$	$F(s-a)$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$f(t-a)u(t-a)$	$e^{-sa}F(s)$
$\delta(t-a)$	e^{-as}

Numerics

- Newton's method: $x_{k+1} = x_k - f(x_k)/f'(x_k)$.
- Newton's method for systems: $\mathbf{J}^{(k)}\left(\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\right) = -\mathbf{f}(\mathbf{x}^{(k)})$ with $(\mathbf{J}^{(k)})_{ij} = \partial_j f_i^{(k)}$
- Lagrange interpolation polynomial: $L_k(x) = \frac{(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$, $p_n(x) = \sum_{k=0}^{n} L_k(x) f(x_k)$
- Trapezoid rule: $\int_a^b f(x)\,dx \approx h\left[\frac{1}{2}f(a) + f(x_1) + f(x_2) + \ldots + f(x_{n-1}) + \frac{1}{2}f(b)\right]$

  Error of the trapezoid rule: $|\epsilon| \le h^2 \frac{b-a}{12} \max_{a\le x\le b} |f''(x)|$.
- Simpson rule: $\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + \ldots + 2f_{n-2} + 4f_{n-1} + f_n\right]$ with $f_i = f(x_i)$.

  Error of the Simpson rule: $|\epsilon| \le h^4 \frac{b-a}{180} \max_{a\le x\le b} |f^{(4)}(x)|$.
- Gauss–Seidel iteration: $\mathbf{x}^{(k+1)} = \mathbf{b} - \mathbf{L}\mathbf{x}^{(k+1)} - \mathbf{U}\mathbf{x}^{(k)}$ with $\mathbf{A} = \mathbf{I} + \mathbf{L} + \mathbf{U}$.
- Jacobi iteration: $\mathbf{x}^{(k+1)} = \mathbf{b} + (\mathbf{I} - \mathbf{A})\mathbf{x}^{(k)}$
- Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_n, \mathbf{y}_n)$
- Improved Euler method: $\mathbf{k}_1 = h\mathbf{f}(x_n, \mathbf{y}_n)$, $\mathbf{k}_2 = h\mathbf{f}(x_n + h, \mathbf{y}_n + \mathbf{k}_1)$,

  $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}\mathbf{k}_1 + \frac{1}{2}\mathbf{k}_2$.
- Classical Runge–Kutte method:

  $\mathbf{k}_1 = h\mathbf{f}(x_n, \mathbf{y}_n)$, $\mathbf{k}_2 = h\mathbf{f}(x_n + h/2, \mathbf{y}_n + \mathbf{k}_1/2)$,

  $\mathbf{k}_3 = h\mathbf{f}(x_n + h/2, \mathbf{y}_n + \mathbf{k}_2/2)$, $\mathbf{k}_4 = h\mathbf{f}(x_n + h, \mathbf{y}_n + \mathbf{k}_3)$,

  $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{6}\mathbf{k}_1 + \frac{1}{3}\mathbf{k}_2 + \frac{1}{3}\mathbf{k}_3 + \frac{1}{6}\mathbf{k}_4$.
- Backward Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_{n+1}, \mathbf{y}_{n+1})$
- Finite differences: $\frac{\partial u}{\partial x}(x,y) \approx \frac{u(x+h,y) - u(x-h,y)}{2h}$

  $\frac{\partial^2 u}{\partial x^2}(x,y) \approx \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2}$

  $\frac{\partial u}{\partial y}(x,y) \approx \frac{u(x,y+h) - u(x,y-h)}{2h}$

  $\frac{\partial^2 u}{\partial y^2}(x,y) \approx \frac{u(x,y+h) - 2u(x,y) + u(x,y-h)}{h^2}$