Miscellaneous

- **Heaviside function** $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$, $u(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases}$
- **Dirac Delta function** $\delta(t-a)$ is zero except at $t = a$, $\int_{-\infty}^{\infty} \delta(t-a)dt = 1$, and $\int_{-\infty}^{\infty} g(t)\delta(t-a)dt = g(a)$ for any continuous function g.
- **Convolution**

  For functions defined on the real line:

$$f * g\,(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy = \int_{-\infty}^{\infty} f(x-y)g(y)dy, \quad x \in \mathbb{R}.$$

  For functions defined only on the positive half-axis:

$$f * g\,(x) = \int_{0}^{x} f(y)g(x-y)dy, \quad x > 0.$$

Laplace transform

- Definition: $\mathcal{L}[f](s) = F(s) = \int_0^{\infty} f(t)e^{-st}dt$

General formulas		$f(t)$	$F(s)$
		1	$\frac{1}{s}$
$\mathcal{L}[e^{at}f(t)](s) = F(s-a)$		$t^n, n = 1, 2, ...$	$\frac{n!}{s^{n+1}}$
$\mathcal{L}[f'](s) = s\mathcal{L}[f] - f(0)$		e^{at}	$\frac{1}{s-a}$
$\mathcal{L}[f''](s) = s^2\mathcal{L}[f] - sf(0) - f'(0)$		$t^n e^{at}, n = 1, 2, ...$	$\frac{n!}{(s-a)^{n+1}}$
$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right](s) = \frac{1}{s}\mathcal{L}[f]$		$\cos bt$	$\frac{s}{s^2+b^2}$
$\mathcal{L}[f * g](s) = \mathcal{L}[f](s)\mathcal{L}[g](s)$		$\sin bt$	$\frac{b}{s^2+b^2}$
$\mathcal{L}[f(t-c)u(t-c)](s) = e^{-cs}F(s),\ c > 0$		$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$\mathcal{L}[tf(t)](s) = -F'(s)$		$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$\mathcal{L}\left[\frac{f(t)}{t}\right](s) = \int_s^\infty F(\sigma)d\sigma$		$u(t-c), c > 0$	$\frac{e^{-cs}}{s}$
		$\delta(t-c), c > 0$	e^{-cs}
		$\cosh bt$	$\frac{s}{s^2-b^2}$
		$\sinh bt$	$\frac{b}{s^2-b^2}$

Fourier series and Fourier transform

- $2L$-periodic functions, real and complex form

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}) \sim \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{in\pi x}{L}}dx$$

- Functions defined on the whole real line (need not be periodic)

$$\hat{f}(w) = \mathcal{F}[f](w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx,$$

$$f(x) = \mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw.$$

- Parseval's identities

$$\frac{1}{2L} \int_{-L}^{L} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2, \qquad \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(w)|^2 dw$$

General formulas		$f(x)$	$\hat{f}(w)$
$\widehat{f'(x)} = iw\hat{f}(w)$		$\delta(x-a)$	$\frac{1}{\sqrt{2\pi}} e^{-iaw}$
$\widehat{f''(x)} = -w^2\hat{f}(w)$		$\begin{cases} 1, & -b \le x \le b \\ 0, & \lvert x \rvert > b \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin bw}{w}$
$\widehat{f(x-a)} = e^{-iaw}\hat{f}(w)$		$e^{-ax}u(x)$	$\frac{1}{\sqrt{2\pi}(a+iw)}$
$\hat{f}(w-b) = \widehat{e^{ibx}f(x)}$		$\frac{1}{x^2+a^2}$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\lvert w \rvert}}{a}$
$\widehat{f*g} = \sqrt{2\pi}\hat{f}\hat{g}$		e^{-ax^2}	$\frac{1}{\sqrt{2a}} e^{-w^2/(4a)}$

Complex numbers and analytic functions

- $e^{x+iy} = e^x(\cos y + i \sin y)$
- $\cos z = \frac{e^{iz}+e^{-iz}}{2}, \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}, \qquad \cosh z = \frac{e^{z}+e^{-z}}{2}, \quad \sinh z = \frac{e^{z}-e^{-z}}{2}$
- Taylor and Laurent series of an analytic function

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz,$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}, \qquad b_n = \frac{1}{2\pi i} \oint_C f(z)(z-z_0)^{n-1} dz$$

Some useful integrals

$\int x \sin ax\, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax + C$

$\int x \cos ax\, dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax + C$

$\int x^2 \sin ax\, dx = \frac{2}{a^2} x \sin ax + \frac{2-a^2x^2}{a^3} \cos ax + C$

$\int x^2 \cos ax\, dx = \frac{2}{a^2} x \cos ax - \frac{2-a^2x^2}{a^3} \sin ax + C$

$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + C$

$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + C$

$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}},\ a > 0$

Some trigonometric identities

$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$

Some important series

- $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$, $\quad \displaystyle\sum_{n=0}^{\infty} x^n$ diverges for $|x| \geq 1$.
- $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ for $x \in \mathbb{R}$.

Linear second order differential equations

Let r_1 and r_2 solve $r^2 + ar + b = 0$. Then

$$y''(x) + ay'(x) + by = 0$$

has general solution given by:

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} \qquad \text{if} \qquad r_1 \neq r_2, \quad r_1, r_2 \in \mathbb{R},$$

$$y(x) = C_1 e^{r_1 x} + C_2 x e^{r_1 x} \qquad \text{if} \qquad r_1 = r_2, \quad r_1, r_2 \in \mathbb{R},$$

$$y(x) = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x) \qquad \text{if} \qquad r_1 = \alpha + i\beta, \quad r_2 = \alpha - i\beta, \quad \alpha, \beta \in \mathbb{R}.$$